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Indefinite derivative Lyapunov–Krasovskii functional method for input to state stability of nonlinear systems with time-delay

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ABSTRACT

In this paper, the problem of the input to state stability (ISS) for nonlinear systems with time-delay is investigated. A continuously differentiable Lyapunov–Krasovskii functional with indefinite derivative is introduced to derive the ISS of the systems, which generalizes the classic Lyapunov–Krasovskii functional with positive definite derivative. As a result, the uniform stability and uniform asymptotic stability criteria of nonlinear systems with time-delay are also established by employing the proposed Lyapunov–Krasovskii functional.

1. Introduction

The notion of input to state stability (ISS) [1] plays a central role in nonlinear systems. The ISS property implies that the states are bounded for bounded inputs, and they tend to the equilibrium of the systems when the inputs tend to zero. Sontag [1] gave a sufficient condition for the ISS property of a nonlinear system by using the Lyapunov function method, and some useful characterizations for ISS systems were obtained in [2–5]. In [6,7], the nonlinear small gain theorem was employed to investigate the ISS property of interconnection of nonlinear systems, and the ISS property of large scale systems and impulsive systems was studied in [8–10], respectively. Later, Dashkovskiy et al. [11,12] studied the integral ISS (iISS) property, which is a nonlinear generalization of $\mathcal{L}^2$ stability and plays the same important role as the ISS property in the analysis and the design of the nonlinear systems.

The phenomenon of time-delay is widespread in practical control systems. The ISS property of nonlinear systems with time-delay has been intensively studied in recently years. The seminal paper [13] showed the nonlinear small-gain theorem was equivalent to the Razumikhin-type theorem for the ISS property of nonlinear systems with time-delay. On the other hand, a Lyapunov–Krasovskii functional method was presented for studying the ISS property of nonlinear systems with time-delay in [14], impulsive systems in [15], and hybrid delayed systems in [16], respectively. Especially in [16], the relationship between the delay bound and the dwell-time bound of the hybrid delayed systems was given detailedly. However, the existing results in [14–16] determining the ISS property of the nonlinear systems with time-delay, needed a Lyapunov–Krasovskii functional with a negative definite derivative, which makes it difficult to choose a suitable one to verify the ISS property.

Recently, an indefinite derivative Lyapunov function method was proposed to investigate the ISS property of the nonlinear delay-free systems in [17], where the ISS property was verified by a positive definite Lyapunov function with an indefinite derivative. It deduces the difficulty of searching a suitable Lyapunov function for an ISS system effectively. Especially, a constructing method of the $\mathcal{KL}$ function is proposed, which plays a very important role in handing the indefinite derivative of the Lyapunov

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function. For nonlinear systems with time-delay, a Lyapunov–Razumikhin function with an indefinite derivative was described to determine the input to state stability in [18], which relaxed the requirement on the derivative of the Lyapunov–Razumikhin function from negative or seminegative definite to indefinite. Another important method investigating the ISS of nonlinear systems with time-delay is the use of the Lyapunov–Krasovskii functional, and the Lyapunov–Razumikhin function method can be considered as a particular case of the method of Lyapunov–Krasovskii functional [14]. We will use a Lyapunov–Krasovskii functional with indefinite derivative to derive the ISS of nonlinear systems with time-delay. Because the derivative of a Lypaunov–Krasovskii functional usually contains the time-delay, and the upper bound of the derivative of the Lypaunov functional is influenced by the current state as well as the past one, therefore, a new Lypaunov–Krasovskii functional including the time-delay should be constructed. In addition, the $\mathcal{KL}$ function presented in [17] is very strict, which may lead to conservativeness.

In this paper, the ISS and the iISS properties of nonlinear systems with time-delay are investigated. To deal with the problem raised by the time-delay, we introduce a new indefinite derivative Lyapunov–Krasovskii functional. By constructing a more general $\mathcal{KL}$ function over [17], some sufficient conditions determining the properties of ISS and iISS are established. As a by-product, some new sufficient conditions for the uniform stability and uniform asymptotic stability are also presented.

This paper is organized as follows. The next section introduces the primary notations and definitions, some Lemmas that will be used in the proof of the main results are also introduced here. Section 3 presents the ISS and the iISS criteria by using an indefinite derivative Lyapunov–Krasovskii functional, some sufficient conditions for uniform stability and asymptotical stability are also derived. Section 4 concludes the paper.

In the following, $\mathbb{R}^+$ denotes the set of all nonnegative real number, $a\wedge b$ denotes the minimum of a and b, $a \vee b$ denotes the maximum of a and b, and $|\cdot|$ denotes the Euclidean norm of a real vector. The space of measurable and local essentially bounded functions is denoted by L_∞ with norm $\|\cdot\|$. $u_{[t_0,t]}(s)$ denotes the truncation of $u(s)$, i.e., if $t_0 \le s \le t$, $u_{[t_0,t]}(s) = u(s)$, while $s > t$, $u_{[t_0,t]}(s) = 0$. $C([-\tau, 0]; \mathbb{R}^n)$ denotes the family of continuous functions ϕ from $[-\tau, 0]$ to R^n with the norm $\|\phi\| = \sup_{-\tau\le s\le 0} |\phi(s)|$. $L^p([-\tau, 0], \mathbb{R}^n)$ denotes the family of all Borel measurable $\mathbb{R}^n$-valued functions $\phi(s)$ defined on $-\tau \le s \le 0$ with norm $\phi = (\int_{-\tau}^0 |\phi(s)|^p ds)^{1/p} < \infty$.

A function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{K}$-function which is denoted by $\gamma \in \mathcal{K}$ if γ is continuous and strictly increasing with $\gamma(0) = 0$, a $\mathcal{K}_\infty$-function which is denoted by $\gamma \in \mathcal{K}_\infty$ if γ is a $\mathcal{K}$-function and satisfies $\lim_{t\to\infty} \gamma(t) = \infty$. A function $\sigma(s, t) : R^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{KL}$-function which is denoted by $\sigma \in \mathcal{KL}$ if it is a $\mathcal{K}$-function for fixed t, and the mapping $\sigma(s, t)$ decreases to zero as $t \to \infty$, for fixed s. For notational simplicity, ISS denotes both input to state stability and input to state stable.

2. Preliminaries

Consider the following nonlinear control system with a time-delay:

$$\dot{x}(t) = f(t, x_t, u(t)) \tag{1}$$

where $x : \mathbb{R} \to \mathbb{R}^n$, and input $u : \mathbb{R}^+ \to \mathbb{R}^m$ are assumed to be measurable and locally essentially bounded; for $t \ge t_0, x_t : [-\tau, 0] \to \mathbb{R}^n$ is given by $x_t(s) = x(t + s)$, τ is the time-delay; $f : \mathbb{R}^+ \times C([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be locally Lipschitz in (t, x) and uniform continuous in u, and to satisfy $f(t, 0, 0) = 0$. Then for a given initial state $x_{t_0} \in C([-\tau, 0]; \mathbb{R}^n)$, and initial time $t_0 \ge 0$, the system (1) exists an unique maximal solution $x(t, x_{t_0}, u)$ in $[t_0, d)$ with $d \in (t_0, \infty]$, and if $d < \infty$, the state $x(t, x_{t_0}, u)$ is unbounded in $[t_0, d)$.

We now give the definitions of ISS and iISS for nonlinear systems with time-delay, which can be found in [14].

Definition 1. The system (1) is said to be ISS if there exist a $\mathcal{KL}$-function $\sigma(s, t)$ and a $\mathcal{K}$-function $\gamma(s)$ such that, for any initial state $x(t_0)$ and any measurable, locally essentially bounded input $u(t)$, the solution exists for all $t \ge t_0$ and satisfies

$$|x(t)| \le \sigma(\|x_{t_0}\|, t - t_0) + \gamma(\|u_{[t_0,t]}\|) \tag{2}$$

Definition 2. System (1) is said to be iISS if there exist a $\mathcal{KL}$-function $\sigma(s, t)$, a $\mathcal{K}_\infty$-function $\alpha(s)$, and a $\mathcal{K}$-function $\gamma(s)$, such that, for any initial state $x(t_0)$, any measurable, locally essentially bounded input $u(t)$, the solution exists for all $t \ge t_0$ and satisfies

$$\alpha(|x(t)|) \le \sigma(\|x_{t_0}\|, t - t_0) + \int_{t_0}^{t} \gamma(|u(\tau)|)d\tau. \tag{3}$$

If $V : \mathbb{R}^+ \times C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^+$ is a continuous function and $x(t, t_0, \xi)$ is the solution of system (1), its derivative is defined as

$$\dot{V}(t, \varphi) = \limsup_{s\to 0^+} \frac{1}{s}[V(t + s, x_{t+s}(t, \varphi)) - V(t, \varphi)]. \tag{4}$$

The following comparison principle, which was proposed in [17], is useful to estimate the bound of a differential inequality with indefinite upper bound. It plays a critical role to prove the ISS property based on an indefinite derivative Lyapunov–Krasovskii functional.

Lemma 1 ([17]). *Suppose that $y : \mathbb{R}^+ \to \mathbb{R}^+$ is an absolutely continuous function; $u : \mathbb{R}^+ \to \mathbb{R}^m$ is a measurable, locally essentially bounded mapping; $\phi : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function; and $\rho \in \mathcal{K}$. If for almost all $t \ge t_0$,*

$$\dot{y}(t) \le \phi(t)y(t), \quad \forall y(t) \ge \rho(|u(t)|)$$