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On solution of functional integral equation of fractional order

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ABSTRACT

The aim of this paper is to investigate existence and stability of the solution of the functional integral equations of fractional order arising in physics, mechanics and chemical reactions. These equations are considered in the Banach space of real functions defined, continuous and bounded on an unbounded interval $\mathbb{R}_+$. The main tools used in our considerations are the concept of a measure of noncompactness and the classical Schauder fixed point theorem. Also, the numerical method is employed successfully for solving these functional integral equations of fractional order.

1. Introduction

Integral equations of fractional order play a very significant role in describing some real world problems. For example some problems in physics, mechanics, engineering, electrochemistry, and other fields can be described with the help of integral equations of fractional order [1–7] and several research papers studied those integral equations [8,9]. Despite many researches in the recent years in the area of solvability of an integral equation of fractional order on a bounded interval, much less has been done on functional integral equations of fractional order on an unbounded interval.

There is an increasing demand for studying functional integral equation of fractional order and these problems of course cannot be solved explicitly. Therefore, it is important to investigate the problem of the existence of solutions of these integral equations. The stability of solutions for fractional differential equations has been studied by many authors, e.g. Deng [10] and Li et al. [11]. Let us pay attention to the fact that only a few papers investigated the stability of solutions for integral equation of fractional order and functional type has been considered less.

Definition 1. Let $x \in L^1(a, b), 0 \le a < b < \infty$, and let $\alpha > 0$ be a real number. The (Riemann–Liouville) fractional integral of order α is defined by

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}} ds, \quad t \in (a, b)$$

where $\Gamma(\alpha)$ denotes the gamma function.

It may be shown that the fractional integral operator I^α transforms the space $L^1(a, b)$ into itself and has some other properties [4,12–14].

Let us notice that the above definition can be easily extended to the space $L^1(a, \infty)$. More generally, we can consider the operator I^α on the function space $L^1_{loc}(a, \infty)$ consisting of real functions being locally integrable over the interval $[a, \infty)$.

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In this paper, we will consider singular functional integral equation of the form

$$x(t) = g(t) + f\left(t, \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\mathrm{d}s, x(h(t,x(t)))\right), \tag{1}$$

where $\alpha \in (0,1)$, $t \in \mathbb{R}_+$, and we have $g(t) \in BC(\mathbb{R}_+)$, $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ and $h : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}_+$ are functions satisfies special assumptions, see Section 3. The singular functional integral equations having the form (1) are also called equations of fractional order since the term with the integral appearing in Eq. (1) can be treated from the viewpoint of the concept of Riemann–Liouville integral of fractional order.

This paper is devoted to study the existence and stability of solution of Eq. (1) in the Banach space of real functions defined, continuous and bounded on an unbounded interval $\mathbb{R}_+$. The technique used here is the measure of noncompactness associated with the Schauder fixed point theorem. Also, This article investigates the numerical scheme based on the Sinc approximation with the single exponential (SE) transformation for solving Eq. (1). In comparison with other bases, it advantages are the exponential convergence of an approximate solution and implementation and accurate approximation, even in the presence of singularities.

The paper is organized as follows. In Section 2, we review some basic facts which will be needed in our further considerations. Section 3 is devoted to study the solvability of the functional integral (1). In Section 4, we investigate the stability of solutions of Eq. (1). In Section 5, Sinc method is introduced to approximate the solution of Eq. (1). In the last section, some examples are given to demonstrate the applicability of our results.

2. Preliminaries

The goal of this section is to recall notations and definitions which will be needed in our further investigations.

Let E be a real Banach space with the norm $\|.\|$ and the zero element θ. Denote by $B(x, r)$ the closed ball centered at x and with radius r. The ball $B(\theta, r)$ will be denoted by B_r. If X is a subset of E then the symbols $\bar{X}$ and $Conv\,X$ stand for the closure and convex closure of X, respectively. We use the standard notation $X + Y$, λX to denote the usual algebraic operations on subsets X and Y of the space E. Further, let $\mathfrak{M}_E$ denote the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

We will accept the following definition about a measure of noncompactness [15].

Definition 2. A mapping $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+$ is said to be a measure of noncompactness in the space E if it satisfies the following conditions:

(1) The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.
(2) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.
(3) $\mu(X) = \mu(\bar{X}) = \mu(ConvX)$.
(4) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$.
(5) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, \dots)$ and if $\lim_{n\to\infty}\mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ defined in (1) is called the kernel of the measure of noncompactness μ.

Remark 1. Let us notice that the intersection set X_∞ described in axiom (5) is a member of the kernel of the measure of noncompactness μ. In fact, the inequality $\mu(X_\infty) \le \mu(X_n)$ for $n = 1, 2, \dots$ implies that $\mu(X_\infty) = 0$. Hence $X_\infty \in \ker\mu$. This property of the set X_∞ will be very important in our investigations.

For further facts concerning measures of noncompactness and their properties we refer the reader to [15].

In what follows, we will work in the Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, continuous and bounded on $\mathbb{R}_+$. This space is furnished with the standard norm

$$\|x\| = \sup\{|x(t)| : t \ge 0\}.$$

We will use a measure of noncompactness in the space $BC(\mathbb{R}_+)$ which was introduced in [15]. In order to define this measure let us fix a nonempty bounded subset X of the space $BC(\mathbb{R}_+)$ and a positive number T. For $x \in X$ and $\varepsilon \ge 0$ denote by $\omega^T(x, \varepsilon)$ the modulus of continuity of the function x on the interval [0, T], i.e.

$$\omega^T(x,\varepsilon) = \sup\{|x(t) - x(s)| \ : \ t, s \in [0,T]\,,\ |t-s| \le \varepsilon\}.$$

Moreover, let us put

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) \ : \ x \in X\},$$
$$\omega_0^T(X) = \lim_{\varepsilon\to 0}\omega^T(X,\varepsilon),$$
$$\omega_0(X) = \lim_{T\to\infty}\omega_0^T(X).$$

If t is a fixed number from $\mathbb{R}_+$, let us denote

$$X(t) = \{x(t) \ : \ x \in X\}$$