Applied Mathematics and Computation 270 (2015) 665–680

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Asymptotic formulas for gamma function with applications

Zhen-Hang Yang, Yu-Ming Chu*

School of Mathematics and Computation Sciences, Hunan City University, Yiyang 413000, China

ARTICLE INFO

MSC:
33B15
26E60
33F05
26D07

Keywords:
Stirling's formula
Gamma function
Bivariate mean
Factorial

ABSTRACT

In the article, we present several asymptotic formulas for the gamma function in terms of the bivariate means. As applications, some sharp upper and lower bounds for the gamma function and factorial n are given.

1. Introduction

It is well known that the Stirling's formula

$$n! \sim \sqrt{2\pi n} n^n e^{-n} \quad (n \to \infty)$$

has important applications in statistical physics, probability theory, number theory, combinatorics and other related fields. Recently, the generalizations and improvements of Stirling's formula have attracted the attention of many researchers.

In [1], Burnside proved that

$$n! \sim \sqrt{2\pi}\left(\frac{n+1/2}{e}\right)^{n+1/2} \quad (n \to \infty).$$

Gosper [2] found that

$$n! \sim \sqrt{2\pi\left(n+\frac{1}{6}\right)}\left(\frac{n}{e}\right)^n \quad (n \to \infty).$$

In [3], Batir obtained an asymptotic formula as follows:

$$n! \sim \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1/6}} \quad (n \to \infty).$$

The following more accurate approximation for $n!$

$$n! \sim \sqrt{2\pi}\left(\frac{n^2+n+1/6}{e^2}\right)^{n/2+1/4} \quad (n \to \infty)$$

can be found in the literature [4].

* Corresponding author. Tel.: +86 572 2321510; fax: +86 572 2321163.
E-mail addresses: yzhkm@163.com (Z.-H. Yang), chuyuming@hutc.zj.cn, chuyuming2005@126.com (Y.-M. Chu).

http://dx.doi.org/10.1016/j.amc.2015.08.087
0096-3003/

Let $x > 0$. Then the classical Euler's gamma function Γ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,$$

and its logarithmic derivative $\psi(x) = \Gamma'(x)/\Gamma(x)$ is known as the psi or digamma function, while ψ', ψ'', ... are called polygamma functions (see [5]).

It is well known that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$. Therefore, to find the asymptotic formulas or bounds for the gamma function also attracted the attention of mathematicians. For example, Ramanujan [6, p. 339] found that the double inequality

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{100}\right)^{1/6} < \Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6}$$

holds for all $x \geq 1$. Batir [7] proved that the double inequality

$$\sqrt{2}e^{4/9}\left(\frac{x}{e}\right)^x \sqrt{x + \frac{1}{2}} \exp\left(-\frac{1}{6(x+3/8)}\right) < \Gamma(x+1) < \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x + \frac{1}{2}} \exp\left(-\frac{1}{6(x+3/8)}\right)$$

holds for all $x > 0$. Mortici [8] proved that

$$\sqrt{2\pi e}e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+1/2} < \Gamma(x+1) \leq \alpha\sqrt{2\pi e}e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+1/2},$$

$$\beta\sqrt{2\pi e}e^{-\varsigma}\left(\frac{x+\varsigma}{e}\right)^{x+1/2} \leq \Gamma(x+1) < \sqrt{2\pi e}e^{-\varsigma}\left(\frac{x+\varsigma}{e}\right)^{x+1/2}$$

for $x \geq 0$ with $\omega = \left(3 - \sqrt{3}\right)/6$, $\alpha = 1.072042464\cdots$, $\varsigma = \left(3 + \sqrt{3}\right)/6$ and $\beta = 0.988503589\cdots$.

More results involving the asymptotic formulas or bounds for $n!$ or gamma function can be found in the literature [9–16] and the references cited therein.

Mortici [17] presented an idea that by replacing an under-approximation and an upper-approximation of the factorial function by one of their geometric mean to improve certain approximation formula of the factorial.

The main purpose of this paper is to get the asymptotic formulas for the gamma function in terms of the bivariate means, and present the sharp bounds for gamma function and $n!$.

2. Main results

A bivariate real valued function M: $(0, \infty) \times (0, \infty) \to (0, \infty)$ is said to be a mean if

$$\min(a, b) \leq M(a, b) \leq \max(a, b)$$

for all $a, b \in (0, \infty)$. Clearly, each bivariate mean M is reflexive, that is,

$$M(a, a) = a$$

for any $a \in (0, \infty)$. M is symmetric if

$$M(a, b) = M(b, a)$$

for all $a, b \in (0, \infty)$, and M is said to be homogeneous (of degree one) if

$$M(ta, tb) = tM(a, b) \tag{2.1}$$

for any $a, b \in (0, \infty)$ and $t > 0$.

Lemma 1 ([18, Theorems 1–3]). *Let $c \in (0, \infty)$ and M: $(0, \infty) \times (0, \infty) \to (0, \infty)$ be a differentiable mean. Then*

$$M_1'(c, c), M_2'(c, c) \in [0, 1], \quad M_1'(c, c) + M_2'(c, c) = 1.$$

In particular, if M is symmetric, then

$$M_1'(c, c) = M_2'(c, c) = \frac{1}{2}.$$

Lemma 2 ([19, Lemma 3.2], [20, Lemma 2]). *Let M: $(0, \infty) \times (0, \infty) \to (0, \infty)$ be a homogeneous and twice differentiable mean. Then the identities*

$$aM_{11}''(a, b) + bM_{12}''(a, b) = 0, \quad aM_{12}''(a, b) + bM_{22}''(a, b) = 0$$

hold for all $a, b \in (0, \infty)$.

Theorem 1. *Let $\theta, \theta^*, \sigma, \sigma^*$ be four fixed real numbers such that $\theta + \theta^* = \sigma + \sigma^* = 1$, and M, N: $(0, \infty) \times (0, \infty) \to (0, \infty)$ be two symmetric, homogeneous and differentiable means. Then*

$$\Gamma(x+1) \sim \sqrt{2\pi} M(x+\theta, x+\theta^*)^{x+1/2} e^{-N(x+\sigma, x+\sigma^*)} \quad (x \to \infty).$$