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S-asymptotically ω-positive periodic solutions for a class of neutral fractional differential equations[☆]

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ABSTRACT

In this paper, we investigate the existence of the S-asymptotically ω-positive periodic solutions to a class of semilinear neutral Caputo fractional differential equations with infinite delay, given by

$$\begin{cases} D_t^{\alpha}(x(t)+F(t,x_t))+A(x(t))=G(t,x_t),\ t\geq 0, \\ x(0)=\varphi\in\mathscr{B}. \end{cases}$$

The function is considered in a Banach space X for $0<\alpha<1$. Here $-A$ denotes the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$.

1. Introduction

The properties of periodic solutions to functional differential equations, integral equations and partial differential equations have been extensively studied. Recently, such study was extended to the almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic, and pseudo-almost periodic solutions to fractional differential equations (see [1–16]). Due to the structures of such equations, investigating their solutions is challenging.

For example, the asymptotic periodicity for some evolution equations in Banach space was studied in [1]. The authors considered a class of semi-linear integral equations with infinite delay of the form

$$u(t)=\int_{-\infty}^{t}a(t-s)[Au(s)+f(s,u(s))]ds,\ t\in R,$$

and obtained a series of sufficient conditions, which guarantees the existence and uniqueness of a weighted pseudo-almost periodic (mild) solution to such equations. Besides this study, [1] also investigated the asymptotically ω-periodic mild solutions to the following fractional differential equations:

$$\begin{cases} \frac{d}{dt}D(t,u_t)=\int_{0}^{t}a(t-s)AD(s,u_s)ds+F(t,u_t),\ t\geq 0 \\ u_0=\psi\in\mathscr{B}, \end{cases}$$

with some conditions under which the equations have a unique asymptotically ω-periodic mild solution derived.

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The positive solutions of fractional differential equations have just recently been studied [23,24]. However, as far as we know, investigating the S-asymptotically ω-positive periodic solutions of fractional differential equations has not been attempted.

In this paper, we investigate the S-asymptotically ω-positive periodic solutions to the following fractional partial neutral differential equations:

$$\begin{cases} D_t^{\alpha}(x(t)+F(t,x_t))+A(x(t))=G(t,x_t),\ t\geq 0, \\ x(0)=\varphi\in\mathscr{B}, \end{cases} \tag{1.1}$$

where D_t^{α} is the Caputo fractional derivative with $0<\alpha<1$ and $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$.

2. Preliminaries

In this section, we state some notations and notions which will be used throughout this article.

2.1. Definitions and Lemmas

In this article, we use $C_b([0,\infty),X)$ to denote the space consisting of the continuous and bounded functions from $[0,\infty)$ into X, endowed with the norm of the uniform convergence, expressed as $\|\cdot\|_{\infty}$. Letting P be a cone in X, we then define a partial ordering in X by $x\leq y$ if and only if $y-x\in P$. On the other hand, if $x\leq y$ and $x\neq y$, we say that $x<y$. P is called normal if there exists a positive constant N such that $\theta\leq x\leq y$, which indicates that $\|x\|\leq N\|y\|$. Here θ is the zero element of X.

Next, we consider some Definitions.

Definition 2.1 ([25]). A function $f\in C_b([0,\infty),X)$ is said to be S-asymptotically periodic if there exists $\omega>0$ such that $\lim_{t\to\infty}(f(t+\omega)-f(t))=0$. For this case, we say that ω is an asymptotic period of f and f is S-asymptotically ω-periodic.

Definition 2.2 ([25]). A continuous function f: $[0,\infty)\times X\to X$ is called uniformly S-asymptotically ω-periodic on bounded sets if, for every bounded subset K of X, the set $\{f(t,x)$: $t\geq 0, x\in K\}$ is bounded and $\lim_{t\to\infty}(f(t,x)-f(t+\omega,x))=0$ uniformly in $x\in K$.

Definition 2.3 ([25]). A continuous function f: $[0,\infty)\times X\to X$ is called asymptotically uniformly continuous on bounded sets if for every $\epsilon>0$ and every bounded subset K of X, there exist $L_{\epsilon,K}\geq 0$ and $\delta_{\epsilon,K}>0$ such that $\|f(t,x)-f(t,y)\|\leq\epsilon$, for all $t\geq L_{\epsilon,K}$ and all $x,y\in K$ with $\|x-y\|\leq\delta_{\epsilon,K}$.

Lemma 2.1 ([16]). *Let f: $[0,\infty)\times X\to X$ be uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets and let u: $[0,\infty)\to X$ be an S-asymptotically ω-periodic function. Then the function $\upsilon(t)=f(t,u(t))$ is S-asymptotically $\omega-$ periodic.*

Definition 2.4 ([26]). Let $a,\alpha\in\mathbb{R}$. A function f: $[a,\infty)\to X$ is said to be in the space $C_{a,\alpha}$ if there exist a real number $p>\alpha$ and a function $g\in C([a,\infty),X)$ such that $f(t)=t^pg(t)$. In addition, f is said to be in the space $C^m_{a,\alpha}$ for some positive integer m if $f^{(m)}\in C_{a,\alpha}$.

Definition 2.5 ([26]). If the function $f\in C^m_{a,\alpha}$ and $m\in\mathbb{N}^+$, then the fractional derivative of order $\alpha>0$ of f in the Caputo sense is defined as

$$D_t^{\alpha}f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-s)^{m-\alpha-1}f^{(m)}(s)ds,\quad m-1<\alpha\leq m.$$

Definition 2.6 ([27]). Let $A:\mathscr{D}\subseteq X\to X$ be a closed linear operator. A is said to be sectorial if there exist $0<\theta<\pi/2$, $M>0$ and $\mu\in\mathbb{R}$ such that the resolvent of A exists outside the sector

$$\mu+S_{\theta}=\{\mu+\lambda:\lambda\in\mathbb{C},|\arg(-\lambda)|<\theta\}$$

and

$$\|(\lambda I-A)^{-1}\|\leq\frac{M}{|\lambda-\mu|},\quad\lambda\notin\mu+S_{\theta}$$

(for short, we say that A is sectorial of type (M,θ,μ)).

If we use $-A$ to denote the infinitesimal generator of an analytic semigroup in a Banach space and use $\rho(A)$ to denote the resolvent set of A, where $0\in\rho(A)$, then the fractional power A^{-q} can be defined as follows:

$$A^{-q}=\frac{1}{\Gamma(q)}\int_0^{\infty}(t)^{q-1}T(t)dt,\quad q>0.$$

We note that A^q is a closed linear operator with domain $D(A^q)\supset D(A)$ dense in X for $0<q\leq 1$. Thus, $D(A^q)$ endowed with the graph norm $\|u\|_{D(A)}=\|u\|+\|A^{-q}u\|$, where $u\in D(A^q)$ is a Banach space. Therefore, it follows from A^{-q} being one to one that