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Bicyclic oriented graphs with skew-rank 6[☆]

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ABSTRACT

Let G^σ be an oriented graph and $S(G^\sigma)$ be its skew-adjacency matrix. The skew-rank of G^σ, denoted by $sr(G^\sigma)$, is the rank of $S(G^\sigma)$. In this paper, we characterize all the bicyclic oriented graphs with skew-rank 6. Let G^σ be a bicyclic oriented graph with pendant vertices but no pendant twins. If $sr(G^\sigma) = 6$, then $6 \le |V(G^\sigma)| \le 10$.

1. Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of G is the $n \times n$ symmetric 0–1 matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if the vertices v_i and v_j are adjacent, and $a_{ij} = 0$, otherwise. The *spectrum* of G is defined as the spectrum of $A(G)$. We call $p(G), n(G), \eta(G)$ the numbers of positive, negative and zero eigenvalues in the spectrum of $A(G)$ including multiplicities, respectively. Obviously $p(G) + n(G) + \eta(G) = n$. The *rank* $r(G)$ of G is denoted as the rank of its adjacency matrix. An *oriented graph* G^σ is a digraph which assigns each edge of G a direction σ, and G is called the *underlying graph* of G^σ. Denoted by (u, v) the *arc* of G^σ, with tail u and head v. The *skew-adjacency matrix* associated to the oriented graph G^σ is the $n \times n$ matrix $S(G^\sigma) = (s_{ij})$, where $s_{ij} = 1$ and $s_{ji} = -1$ if (v_i, v_j) is an arc of G^σ, otherwise $s_{ij} = s_{ji} = 0$. The *skew-rank* $sr(G^\sigma)$ of an oriented graph G^σ is defined as the rank of the skew-adjacency matrix $S(G^\sigma)$. Since $S(G^\sigma)$ is skew-symmetric, every eigenvalue of $S(G^\sigma)$ is a pure imaginary number or 0, and the skew-rank of an oriented graph is even.

An *induced subgraph* of G^σ is an induced subgraph of G and each edge preserves the original orientation in G^σ. For a vertex $v \in V(G^\sigma)$, we write $G^\sigma - v$ for the oriented graph obtained from G^σ by removing the vertex v and all edges incident with v. For an induced subgraph H^σ of G^σ, let $G^\sigma - H^\sigma$ be the subgraph obtained from G^σ by deleting all vertices of H^σ and all incident edges. The *degree* of a vertex v for an oriented graph G^σ is the number of the vertices incident to v in its undirected graph G. A vertex of an oriented graph G^σ is called *pendant* vertex if its degree is 1 in G^σ, and is called *quasi-pendant* vertex if it is adjacent to a pendant vertex. Denoted by $K_n, P_n, C_n, K_{1,n-1}$ a complete graph, a path, a cycle and a star all of order n, respectively. A graph is called *trivial* if it has one vertex and no edges, it is sometimes denoted by K_1 or P_1.

Let $C_n^\sigma = v_1v_2, \ldots, v_nv_1$ be an even oriented cycle. The *sign* $\mathrm{sgn}(C_n^\sigma)$ of C_n^σ is defined as the sign of $\prod_{i=1}^{n} s_{v_iv_{i+1}}$ with $v_{n+1} = v_1$. An even oriented cycle C_n^σ is called *evenly-oriented (oddly-oriented)* if its sign is positive (negative). G^σ is called *evenly-oriented* if every even cycle in G^σ is evenly-oriented.

A *bicyclic* graph is a simple connected graph in which the number of edges equals the number of vertices plus one. Let G be a bicyclic graph, the *base* of G is the unique bicyclic subgraph of G containing no pendant vertices. Let $C_p(p \ge 3)$ and $C_q(q \ge 3)$ be

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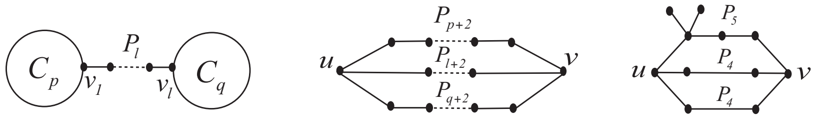

Fig. 1. Graphs $\infty(p, l, q)$, $\theta(p, l, q)$ and G^*.

two vertex-disjoint cycles of length p, q and $P_l = v_1 v_2 \cdots v_l (l \geq 1)$ be a path of length $l-1$. Assume that $v \in V(C_p)$ and $u \in V(C_q)$, let $\infty(p, l, q)$ be the graph obtained from C_p, C_q, P_l by identifying v with v_1 and u with v_l, respectively (as shown in Fig. 1). The bicyclic graph containing $\infty(p, l, q)$ as its base is called an *∞-graph*.

Let $P_{p+2}, P_{l+2}, P_{q+2}$ be three paths with $\min\{p, l, q\} \geq 0$ and at most one of p, l, q is 0. Let $\theta(p, l, q)$ be the graph obtained from $P_{p+2}, P_{l+2}, P_{q+2}$ by identifying the three initial vertices and terminal vertices (as shown in Fig. 1). The bicyclic graph containing $\theta(p, l, q)$ as its base is called a *θ-graph*.

Recently the skew-adjacency matrix of an oriented graph has received a lot of attentions. Gutman introduced the energy of a simple undirected graph in [10]. Several results on the energy of the adjacency matrix of a graph have been obtained in [7,9,11,12,17,21] and the book [16]. Recently more concepts of graph energy are investigated, such as Randić energy [3,8], incidence energy [2,4], Laplacian energy [6], matching energy [14], distance energy [25] for an undirected graph, and skew energy for an oriented graph [1,15] etc. Cavers et al. [5] studied the skew-adjacency matrices of oriented graphs. IMA-ISU research group on minimum rank [13] defined the minimum skew-rank of a simple graph G to be the smallest possible rank among all skew-symmetric matrices over a field F whose ijth entry (for $i \neq j$) is nonzero whenever $\{i, j\}$ is an edge in G and is zero otherwise, and they obtained some results about the minimum skew-rank of graphs. Qu, Yu and Feng [24] obtained more results about the minimum skew-rank of graphs. They also characterized the unicyclic graphs with skew-rank 4 or 6, respectively. Mallik and Shader [22] established some necessary conditions for a graph to have the minimum skew-rank 4 and gave several families of graphs with skew-rank 4. Li and Yu [18] studied the skew-rank of oriented graphs and characterized oriented unicyclic graphs attaining the minimum value of the skew-rank among oriented unicycle graphs of order n with girth k. Qu and Yu [23] characterized the bicyclic oriented graphs with skew-rank 2 or 4.

The rest of this paper is organized as follows: in Section 2, some necessary lemmas are introduced and we point out a small mistake of Theorem 2 of Qu and Yu [23] and give the right conclusion. In Section 3, we characterize all the bicyclic oriented graphs with skew-rank 6, and give a conjecture about the connection between the skew-rank and the number of vertices of a bicyclic oriented graph with pendant vertices but no pendant twins. In Section 4, we give a table to conclusion the results in Theorems 3.1, 3.2 and 3.3.

2. Preliminaries

In this section, we list some elementary lemmas and known results.

Lemma 2.1 ([23]).

(a) *Let H^σ be an induced subgraph of G^σ. Then $sr(H^\sigma) \leq sr(G^\sigma)$.*
(b) *Let $G^\sigma = G_1^\sigma \cup G_2^\sigma \cup \cdots \cup G_t^\sigma$, where $G_1^\sigma, G_2^\sigma, \ldots, G_t^\sigma$ are connected components of G^σ. Then $sr(G^\sigma) = \sum_{i=1}^t sr(G_i^\sigma)$.*
(c) *Let G^σ be an oriented graph on n vertices. Then $sr(G^\sigma) = 0$ if and only if G^σ is an empty graph.*

Lemma 2.2 ([23]). *Let C_n^σ be an oriented cycle of order n. Then we have*

$$sr\left(C_n^\sigma\right) = \begin{cases} n, & C_n^\sigma \text{ is oddly-oriented}, \\ n-2, & C_n^\sigma \text{ is evenly-oriented}, \\ n-1, & \text{otherwise}. \end{cases}$$

Lemma 2.3 ([23]). *Let P_n^σ be an oriented path of order n. Then we have*

$$sr\left(P_n^\sigma\right) = \begin{cases} n-1, & n \text{ is odd}, \\ n, & n \text{ is even}. \end{cases}$$

Lemma 2.4 ([18]). *Let G^σ be an oriented graph containing a pendant vertex, and H^σ be the induced subgraph of G^σ obtained by deleting this pendant vertex together with its neighbor. Then $sr(G^\sigma) = sr(H^\sigma) + 2$.*

Two pendant vertices are called *pendant twins* in G^σ if they have the same neighbor in G^σ.

Lemma 2.5 ([18]). *Let u, v be pendant twins of an oriented graph G^σ. Then we have $sr(G^\sigma) = sr(G^\sigma - u) = sr(G^\sigma - v)$.*

Lemma 2.6 ([19]). *Let C_n^σ be an oriented cycle of order $n (n \geq 3)$ and H^σ be an oriented graph of order $m (m \geq 1)$. Assume that G^σ is the graph obtained by identifying a vertex of C_n^σ with a vertex of H^σ (i.e., $V(C_n^\sigma) \bigcap V(H^\sigma) = v$). Let $B^\sigma = H^\sigma - v$ be the induced subgraph*