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On a nonlinear delay population model

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ABSTRACT

The nonlinear delay differential equation $\dot{x}(t) = r(t)[g(t, x_t) - h(x(t))],\ t \geq 0$ is considered. Sufficient conditions are established for the uniform permanence of the positive solutions of the equation. In several particular cases, explicit formulas are given for the upper and lower limit of the solutions. In some special cases, we give conditions which imply that all solutions have the same asymptotic behavior, in particular, when they converge to a periodic or constant steady-state.

1. Introduction

The scalar nonautonomous differential equation

$$\dot{N}(t) = a(t)N(t) - r(t)N^2(t), \quad t \geq 0 \tag{1.1}$$

is known as the logistic equation in mathematical ecology. Eq. (1.1) is a prototype in modeling the dynamics of single species population systems whose biomass or density is denoted by a function N of the time variable. The functions $a(t)$ and $r(t)$ are time dependent net birth and self-inhibition rate functions, respectively. The carrying capacity of the habitat is the time dependent function

$$K(t) = \frac{a(t)}{r(t)}, \quad t \geq 0. \tag{1.2}$$

By using this notation, Eq. (1.1) can be written as

$$\dot{N}(t) = r(t)(K(t)N(t) - N^2(t)), \quad t \geq 0, \tag{1.3}$$

or

$$\dot{N}(t) = r(t)(K_0 N(t) - N^2(t)), \quad t \geq 0 \tag{1.4}$$

whenever the carrying capacity is constant, i.e., $K(t) = K_0,\ t \geq 0$ with a $K_0 > 0$.

It follows by elementary techniques that the above equations with the initial condition

$$N(0) = N_0 > 0 \tag{1.5}$$

has a unique solution $N(N_0)(t)$ of the initial value problem (IVP) (1.4) and (1.5) given by the explicit formula

$$N(N_0)(t) = \frac{N_0 K_0 e^{K_0 \int_0^t r(s)\,ds}}{K_0 + N_0(e^{K_0 \int_0^t r(s)\,ds} - 1)}, \quad t \geq 0. \tag{1.6}$$

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From the above formula, we get that either

$$\int_0^\infty r(s)\,ds = \infty \tag{1.7}$$

and

$$N(N_0)(\infty) := \lim_{t\to\infty} N(t) = K_0 \quad \text{for any } N_0 > 0,$$

or

$$\int_0^\infty r(s)\,ds < \infty \tag{1.8}$$

and

$$N(N_0)(\infty) = \frac{N_0K_0e^{K_0\int_0^\infty r(s)\,ds}}{K_0 + N_0(e^{K_0\int_0^\infty r(s)\,ds} - 1)} \neq K_0 \quad \text{for any } N_0 \neq K_0.$$

Thus K_0 is a global attractor of (1.4) with respect to the positive solutions if and only (1.7) holds.

It follows by some elementary technique that for any $N_0 > 0$ the solution $N(N_0)(t)$ of the IVP (1.3) and (1.5) obeys

$$\underline{K}(\infty) \le \liminf_{t\to\infty} N(N_0)(t) \le \limsup_{t\to\infty} N(N_0)(t) \le \overline{K}(\infty) \tag{1.9}$$

for any $N_0 > 0$, if

$$0 < \underline{K}(\infty) := \liminf_{t\to\infty} K(t) \le \limsup_{t\to\infty} K(t) =: \overline{K}(\infty) < \infty \tag{1.10}$$

and (1.7) holds. Motivated by the above simple results, in this paper we give lower and upper estimations for the positive solutions of the nonlinear delay differential equation

$$\dot{x}(t) = r(t)(g(t, x_t) - h(x(t))), \quad t \ge 0, \tag{1.11}$$

where $x_t(\theta) = x(t+\theta)$, $-\tau \le \theta \le 0$, $r, h \in C(\mathbb{R}_+, \mathbb{R}_+)$, $g \in C(\mathbb{R}_+ \times C, \mathbb{R}_+)$. Here $\tau > 0$ is fixed, $\mathbb{R}_+ := [0, \infty)$ and $C := C([-\tau, 0], \mathbb{R})$. Eq. (1.11) can be considered as a population model equation with delay in the birth term $r(t)g(t, x_t)$, and no delay in the self-inhibition term $r(t)h(x(t))$. The form of the delay is based on the works of the authors [3,5,8,10–12,14–17], who have argued that the delay should enter in the birth term rather than in death of inhibition term. Eq. (1.11) includes, e.g., the next equations

$$\dot{x}(t) = \sum_{k=1}^n \alpha_k(t)x(t-\tau_k(t)) - \beta(t)x^2(t), \quad t \ge 0, \tag{1.12}$$

$$\dot{x}(t) = \sum_{k=1}^n \alpha_k(t)x^p(t-\tau_k(t)) - \beta(t)x^q(t), \quad t \ge 0, \quad 0 < p < q, \quad q \ge 1, \tag{1.13}$$

$$\dot{x}(t) = \alpha(t)f(x(t-\tau)) - \beta(t)g(x(t)), \quad t \ge 0, \tag{1.14}$$

and

$$\dot{x}(t) = \frac{\alpha(t)x(t-\tau)}{1+\gamma(t)x(t-\tau)} - \beta(t)x^2(t), \quad t \ge 0 \tag{1.15}$$

with discrete delays, or

$$\dot{x}(t) = \alpha(t)\int_{-\tau}^0 f(s, x(t+s))\,ds - \beta(t)g(x(t)), \quad t \ge 0 \tag{1.16}$$

with distributed delay.

Recently, lower and upper estimations of the positive solutions of Eq. (1.12) were proved in [2] and [6] under the assumptions that the coefficients α_k and β satisfy

$$\alpha_0 \le \alpha_k(t) \le A_0, \quad \beta_0 \le \beta(t) \le B_0, \quad t \ge 0, \quad k = 1, \dots, n \tag{1.17}$$

with some positive constants α_0, A_0, β_0 and B_0. The following theorem, which is a consequence of our main results, illustrate that the above boundedness conditions can be released. In this statement we investigate the qualitative behavior of the solution of Eq. (1.12) under the initial condition

$$x(t) = \varphi(t), \quad -\tau \le t \le 0, \tag{1.18}$$

where $\varphi \in C_+ := \{\psi \in C([-\tau, 0], \mathbb{R}_+) : \psi(0) > 0\}$. The unique solution of Eqs. (1.12) and (1.18) is denoted by $x(\varphi)(t)$. We will assume

$$\alpha_k, \tau_k \in C(\mathbb{R}_+, \mathbb{R}_+), \quad (k = 1, \dots, n), \quad \tau := \max_{1\le k\le n} \sup_{t\ge 0} \tau_k(t) < \infty, \tag{1.19}$$