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About analyticity for the coupled system of linear thermoviscoelastic equations

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ABSTRACT

In this paper we consider the classical linear theory of thermoviscoelasticity for inhomogeneous and anisotropic materials in three dimensional space. We show that under suitable conditions, the semigroup associated with the system of the viscoelastic equation of motion coupled with the parabolic equation of energy is analytic.

1. Introduction

We consider the following coupled system of linear thermoviscoelastic equations for inhomogeneous anisotropic materials in three dimensional space

$$\rho\, \mathbf{u}_{tt} = \operatorname{div}(\mathbf{C}[\nabla \mathbf{u}] + \theta\, \mathbf{M}) + \operatorname{div}(\mathbf{C}_1[\nabla \mathbf{u}_t]) \tag{1.1}$$

$$c\,\theta_t = \theta_0\, \mathbf{M} \cdot \nabla \mathbf{u}_t + \operatorname{div}(\mathbf{K}\, \nabla \theta) \tag{1.2}$$

in $\Omega \times (0, +\infty)$ such that $\Omega \subseteq \mathbb{R}^3$ is bounded by the piecewise smooth surface $\partial\Omega \equiv \Gamma$, where $\partial\Omega$ is the boundary of Ω. The motion of the body is referred to the reference configuration and a fixed set of rectangular cartesian axes, relative to which Ω is a rest to the uniform temperature $\theta_0 > 0$. The functions $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $\theta = \theta(\mathbf{x}, t)$ are the displacement and temperature deviation respectively, from the natural state of the reference configuration. Here, $\mathbf{C}$ and $\mathbf{C}_1$ are fourth-order tensors which represent the elasticity and anisotropic viscosity terms, respectively, while $\mathbf{M}$, $\mathbf{K}$, and c are the stress-temperature, conductivity, and specific heat field, respectively. The specific heat c, and the density ρ are prescribed constant fields. The fourth-order tensors $\mathbf{C}$ and $\mathbf{C}_1$ are symmetric; that is, for any pair of symmetric (second-order) tensors $\mathbf{R}$ and $\mathbf{S}$

$$\mathbf{R} \cdot \mathbf{C}[\mathbf{S}] = \mathbf{S} \cdot \mathbf{C}[\mathbf{R}] \quad \text{and} \quad \mathbf{R} \cdot \mathbf{C}_1[\mathbf{S}] = \mathbf{S} \cdot \mathbf{C}_1[\mathbf{R}]. \tag{1.3}$$

The stress-temperature tensor is symmetric, i. e., $\mathbf{M} = \mathbf{M}^T$. Furthermore, we make the common assumption that the conductivity tensor $\mathbf{K}$ is symmetric, i. e., $\mathbf{K} = \mathbf{K}^T$. A further consequence of the heat conduction inequality is the following dissipation inequality:

$$\nabla\theta \cdot \mathbf{K}\, \nabla\theta \geq 0 \quad \text{on} \quad \Omega. \tag{1.4}$$

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Let Γ_1, Γ_2, Γ_3, and Γ_4 be fixed subsets of Γ such that $\overline{\Gamma}_1 \cup \Gamma_2 = \overline{\Gamma}_3 \cup \Gamma_4 = \Gamma$, $\Gamma_1 \cap \Gamma_2 = \Gamma_3 \cap \Gamma_4 = \emptyset$ and $\text{meas}(\Gamma_1) > 0$, $\text{meas}(\Gamma_3) > 0$. We assume that a scalar field $Q(\cdot) \in L^\infty(\Gamma_4)$, $Q(\cdot) > 0$ is assigned on Γ_4.

Notation. ∇ *denotes the gradient operator of a scalar or a vector field,* $\text{div}(\cdot)$ *denotes the divergence operator of a second-order tensor field. We denote by* $\mathbf{n}$ *the outward unit normal, and by dA the element of surface area of A.*

Definition 1.1. By a solution of the mixed initial boundary value problem in $\Omega \times (0, +\infty)$ we mean a pair $(\mathbf{u}, \theta)$ satisfying (1.1)–(1.2) for all $(\mathbf{x}, t) \in \Omega \times (0, +\infty)$, together with boundary conditions:

$$\mathbf{u} = 0 \quad \text{on} \quad \Gamma_1 \times (0, +\infty), \quad (\mathbf{C}[\nabla \mathbf{u}] + \mathbf{C}_1[\nabla \mathbf{u}_t] + \theta\, \mathbf{M}) \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_2 \times (0, +\infty) \tag{1.5}$$

$$\theta = 0 \quad \text{on} \quad \Gamma_3 \times (0, +\infty), \quad \mathbf{K}\,\nabla\theta \cdot \mathbf{n} + Q\,\theta = 0 \quad \text{on} \quad \Gamma_4 \times (0, +\infty) \tag{1.6}$$

and the initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}), \quad \mathbf{u}_t(\mathbf{x}, 0) = \mathbf{u}_t^0(\mathbf{x}), \quad \theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}), \quad x \in \Omega \tag{1.7}$$

where $\mathbf{u}^0(\mathbf{x})$, $\mathbf{u}_t^0(\mathbf{x})$, and θ^0 are prescribed functions in determined spaces. In thermodynamical terms, the first relation in (1.6) states that the part Γ_3 of the boundary is kept at a constant temperature θ_0, while the rest Γ_4 is radiating into a surrounding medium at temperature θ_0.

Remark 1.2. The governing Eqs. (1.1) and (1.2) can be written as

$$\rho\, \frac{\partial^2 u^i}{\partial t^2} = \frac{\partial}{\partial x_j}\left(C_{ijkl}\, \frac{\partial u^k}{\partial x_l} + \theta\, M_{ij}\right) + \frac{\partial}{\partial x_j}\left(C^1_{ijkl}\, \frac{\partial u^k_t}{\partial x_l}\right), \quad i = 1, 2, 3. \tag{1.8}$$

$$c\, \frac{\partial \theta}{\partial t} = \theta_0\, M_{ij}\, \frac{\partial u^i_t}{\partial x_j} + \frac{\partial}{\partial x_i}\left(K_{ij}\, \frac{\partial \theta}{\partial x_j}\right). \tag{1.9}$$

Here we are using the summation convention, summing over repeated indices.

The system (1.1)–(1.2) without the additional anisotropic viscosity term in the Eq. (1.1), was derived by Chiritá [3] in a simpler manner. In particular, the system consists of a hyperbolic equation of motion coupled with the parabolic equation of energy, and it is established the asymptotic behavior of Cesàro means of an energy function and its terms, as $t \to \infty$. Existence, uniqueness, and regularity results were given by Dafermos [4], considering certain additional conditions to the initial boundary value. Other results were established in [5,6,8] and references therein. Other results of stability for thermoviscoelastic models in homogeneous porous media can be seen in [1,2].

The study of mechanical and thermal behavior in anisotropic media has many applications, such as in structural engineering and various industrial processes. In particular, reinforcements and laminations, are the source of anisotropy in the composites. Nondestructive evaluation of composite materials is based on mathematical modeling of wave propagation involving anisotropy and thermoelasticity (see Jiang and Racke [7]). Moreover, anisotropy is observed in nature such as in many types of rocks in geological and tectonic environments (see Sharma [11]). In this paper, we show that adding a viscous term in this quite general model, the analyticity of the associated semigroup is obtained. The consequence is that under certain physically reasonable assumptions, the solution of the thermoviscoelastic system is infinitely smooth even in anisotropic case, and that the energy decays exponentially, being that the zero is on the resolvent [9,10,12].

This paper is organized as follows: Section 2 outlines briefly the notation. In Section 3, the well-posedness of the system is established. In Section 4, we show that the semigroup generated by $\mathcal{A}$ associated to (1.1) and (1.2) is analytic with respect to its domain $\mathcal{D}(\mathcal{A})$, but it does not give any information about the analyticity of the system.

2. Preliminaries

Before beginning with the semigroup setting we present a well-known conservation law of energy in linear thermoelasticity for (1.1) and (1.2). An existence result can be established, combining the arguments given in Dafermos [4].

Theorem 2.1. *Let* $(\mathbf{u}, \theta)$ *be a regular solution of the initial boundary value problem defined by (1.1)–(1.7). Then, the total energy* $\mathcal{E} : \mathbb{R}^+ \to \mathbb{R}^+$ *is given at time t by*

$$\mathcal{E}(t) + \int_0^t \int_\Omega \frac{1}{\theta_0}\, \nabla\theta \cdot \mathbf{K}\,\nabla\theta\, d\mathbf{x}\, ds + \int_0^t \int_\Omega \nabla \mathbf{u}_t \cdot \mathbf{C}_1[\nabla \mathbf{u}_t]\, d\mathbf{x}\, ds + \int_0^t \int_{\Gamma_4} \frac{1}{\theta_0}\, Q\,\theta^2\, d\Gamma_4\, ds = \mathcal{E}(0), \tag{2.1}$$

where

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega \left(\rho\, \mathbf{u}_t \cdot \mathbf{u}_t + \nabla \mathbf{u} \cdot \mathbf{C}[\nabla \mathbf{u}] + \frac{c}{\theta_0}\, \theta^2\right) d\mathbf{x}. \tag{2.2}$$