Applied Mathematics and Computation 270 (2015) 977–983

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Generalization of Jakimovski–Leviatan type Szasz operators

Sezgin Sucu*, Serhan Varma

Ankara University Faculty of Science, Department of Mathematics, Tandoğan TR-06100, Ankara, Turkey

ARTICLE INFO

MSC:
41A10
41A36
33C45

Keywords:
Szasz operator
Modulus of continuity
Rate of convergence
Sheffer polynomials
Meixner polynomials

ABSTRACT

The purpose of this paper is to give a Stancu type generalization of Jakimovski–Leviatan type Szasz operators defined by means of the Sheffer polynomials. We obtain convergence properties of our operators with the help of Korovkin theorem and the order of approximation by using classical and second modulus of continuity. Explicit examples with our operators including Meixner polynomials and the 2-orthogonal polynomials of Laguerre type are given. We present two significant numerical mathematical algorithms as examples for the error estimation.

1. Introduction

In 1950, Szasz [1] introduced the following operators

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \tag{1.1}$$

where $x \geq 0$ and $f \in C[0,\infty)$ whenever the right-hand side of (1.1) exists. In this paper, Szasz investigated the detailed approximation properties of operators (1.1).

Later in 1969, Jakimovski and Leviatan [2] gave a generalization of Szasz operators by using the Appell polynomials. Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ be an analytic function in the disc $|z| < R$, $(R > 1)$ and assume $g(1) \neq 0$. According to Chihara, the Appell polynomials $p_k(x)$ are defined by the generating functions

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k. \tag{1.2}$$

Jakimovski and Leviatan constructed the operators $P_n(f;x)$ by

$$P_n(f;x) := \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)f\left(\frac{k}{n}\right). \tag{1.3}$$

The authors studied the approximation properties of these operators in view of Szasz's method [1]. For the special case $g(z) = 1$, we obtain the Appell polynomials $p_k(x) = \frac{x^k}{k!}$ and from (1.3) we meet again Szasz operators given by (1.1).

* Corresponding author. Tel.: 00903122126720.
E-mail addresses: ssucu@ankara.edu.tr (S. Sucu), svarma@science.ankara.edu.tr (S. Varma).

http://dx.doi.org/10.1016/j.amc.2015.08.077
0096-3003/

Wood [3] proved that the operators given by (1.3) are positive in $[0, \infty)$ if and only if $\frac{a_k}{g(1)} \geq 0$ for $k \in \mathbb{N}$. Taking into account of this fact, Ciupa [4,5] studied the order of approximation of the function f by means of the linear positive operators, introduced another variants of operators (1.3) and examined the approximation properties by virtue of Korovkin's theorem [6]. Several important contributions for Jakimovski–Leviatan type operators can be found in [7–9].

Ismail [10] presented another generalization of Szasz operators (1.1) by using Sheffer polynomials which also include the operators (1.3). Let $A(z) = \sum_{k=0}^{\infty} a_k z^k$, $(a_0 \neq 0)$ and $H(z) = \sum_{k=1}^{\infty} h_k z^k$, $(h_1 \neq 0)$ be analytic functions in the disc $|z| < R$, $(R > 1)$ where a_k and h_k are real. A polynomial set $\{p_k(x)\}_{k \geq 0}$ is said to be Sheffer polynomial set if and only if it has a generating function of the form

$$A(t)e^{xH(t)} = \sum_{k=0}^{\infty} p_k(x)t^k, \quad |t| < R. \tag{1.4}$$

Under the assumptions

$$\begin{aligned} &(i) \text{ for } x \in [0, \infty),\ p_k(x) \geq 0, \\ &(ii)\ A(1) \neq 0 \text{ and } H'(1) = 1, \end{aligned} \tag{1.5}$$

Ismail investigated the approximation properties of the following

$$T_n(f; x) := \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad \text{for } n > 0. \tag{1.6}$$

For $H(t) = t$, one can easily obtain that the operators given by (1.6) reduce to the operators (1.3). On one hand, by choosing $H(t) = t$ and $A(t) = 1$, we get $p_k(x) = \frac{x^k}{k!}$, therefore (1.6) leads to the well-known Szasz operators (1.1).

In this paper, under the assumptions (1.5), we define a Stancu type generalization of the operators (1.6) as below:

$$S_n^{\alpha,\beta}(f; x) := \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k+\alpha}{n+\beta}\right), \quad \text{for } n > 0 \tag{1.7}$$

where $\alpha, \beta \geq 0$. Convergence of the operators (1.7) is examined with the help of the well-known Korovkin theorem. The degree of convergence is established by using classical and the second modulus of continuity. Operators including Meixner polynomials which form a sequence of discrete orthogonal polynomials and the 2-orthogonal polynomials of Laguerre type are given as examples. In addition, numerical examples are presented to illustrate the theoretical results. It is worthy to note that when $\alpha = \beta = 0$, we find the operators (1.6).

2. Approximation properties of $S_n^{\alpha,\beta}$ operators

We give the following lemmas and definitions which are used in the sequel.

Definition 1. Let $f \in \tilde{C}[0, \infty)$ and $\delta > 0$. The modulus of continuity $\omega(f; \delta)$ of the function f is defined by

$$\omega(f; \delta) := \sup_{\substack{x,y \in [0,\infty) \\ |x-y| \leq \delta}} |f(x) - f(y)|,$$

where $\tilde{C}[0, \infty)$ is the space of uniformly continuous functions on $[0, \infty)$.

Definition 2. The second modulus of continuity of the function $f \in C[a, b]$ is defined by

$$\omega_2(f; \delta) := \sup_{0<t\leq\delta} \|f(.+2t) - 2f(.+t) + f(.)\|,$$

where $\|f\| = \max_{x \in [a,b]} |f(x)|$.

Lemma 1. *(Gavrea and Rasa [11]) Let $z \in C^2[0, a]$ and $(L_n)_{n \geq 0}$ be a sequence of linear positive operators with the property $L_n(e_0; x) = e_0(x)$, $e_i(\xi) = \xi^i$, $i \in \{0, 1, 2\}$. Then*

$$|L_n(z; x) - z(x)| \leq \|z'\| \sqrt{L_n((\xi - x)^2; x)} + \frac{1}{2}\|z''\| L_n((\xi - x)^2; x). \tag{2.1}$$

Lemma 2. *(Zhuk [12]) Let $f \in C[a, b]$ and $h \in \left(0, \frac{b-a}{2}\right)$. Let f_h be the second-order Steklov function attached to the function f. Then the following inequalities are satisfied*

$$\begin{aligned} &(i)\ \|f_h - f\| \leq \tfrac{3}{4}\omega_2(f; h) \\ &(ii)\ \|f_h''\| \leq \tfrac{3}{2h^2}\omega_2(f; h). \end{aligned} \tag{2.2}$$