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Adams method for solving uncertain differential equations

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ABSTRACT

For uncertain differential equations, we cannot always obtain their analytic solutions. Early researchers have described the Euler method and Runge–Kutta method for solving uncertain differential equations. This paper proposes a new numerical method—Adams method to solve uncertain differential equations. Some numerical experiments are given to illustrate the efficiency of our numerical method. Moreover, this paper also gives two numerical methods for calculating the extreme value and the time integral of solutions of uncertain differential equations.

1. Introduction

In order to model nondeterministic phenomena, two mathematical systems are used, among others: probability theory that is based on the concept of frequency, while another is uncertainty theory based on the degree of belief. Before applying probability theory in practice, we should obtain a probability distribution that is close enough to the real frequency via statistics. Otherwise, we have to ask domain experts to estimate their degree of belief that each event will happen. Through many surveys, Kahneman and Tversky [5] showed that *human beings usually overweight unlikely events*. From another point of view, Liu [12] showed that *human beings usually estimate a much wider range of values than the object actually takes*. This conservatism of human beings makes the degree of belief deviate far from the frequency. If we still consider human degrees of belief as probability distributions, then we may obtain counterintuitive results (Liu [10]).

Since Liu [6,9] established uncertainty theory and refined it, uncertainty theory has become a new branch of mathematics for modeling nondeterministic phenomena. For a more detailed exposition of uncertainty theory with applications, the readers may consult the recent book [12]. In the framework of uncertainty theory, Liu [8] proposed a canonical Liu process that is a Lipschitz continuous uncertain process with stationary independent increments and increments are normal uncertain variables. In addition, Liu [8] introduced uncertain calculus to deal with the differentiation and integration of uncertain process. Uncertain differential equations were studied by Liu [7], as a type of differential equations driven by a canonical Liu process. Up to now, uncertain differential equations have been successfully applied in practice.

Chen and Liu [1] proved the existence and uniqueness theorem for the solution of an uncertain differential equation under linear growth condition and Lipschitz continuous condition. Gao [3] verified this theorem under local linear growth condition and local Lipschitz continuous condition. Liu [8] introduced the concept of stability of an uncertain differential equation, and some stability theorems were proved by Yao et al. [21]. Based on this work, different types of stability were extended, for example, stability in mean (Yao et al. [24]); stability in moment (Sheng and Wang [16]); almost sure stability (Liu et al. [13]); and exponential stability (Sheng and Gao [17]).

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Chen and Liu [1] studied an analytic solution for linear uncertain differential equations. Liu [14] and Yao [23] considered a spectrum of analytic methods to solve some special classes of nonlinear uncertain differential equation. However, for general uncertain differential equations, we cannot obtain their analytic solutions. Thus, it is important to design some numerical methods to solve uncertain differential equations. Yao and Chen [20] proved that the solution of an uncertain differential equation can be represented by a spectrum of ordinary differential equations. And they designed the Euler method to solve the uncertain differential equation. Later, Yang and Shen [19] designed the Runge–Kutta method to solve the uncertain differential equation. Uncertain differential equations have been widely applied in many fields such as uncertain finance (Liu [11]), uncertain optimal control (Zhu [25]), and uncertain differential game (Yang and Gao [18]).

In this paper, we will present Adams method for solving uncertain differential equations. The rest of the paper is arranged as follows. Section 2 reviews some basic concepts and properties of uncertain differential equations. Section 3 studies a new numerical method—Adams method—to solve an uncertain differential equation. Sections 4 and 5 propose two numerical methods to calculate the extreme value and the time integral of solutions of uncertain differential equations, respectively.

2. Uncertain differential equation

An uncertain measure $\mathcal{M}$ is a real-valued set-function on a σ-algebra $\mathcal{L}$ over a nonempty set Γ which satisfies normality, duality, subadditivity and product axioms. For more details, see Appendix A. In this section, we will introduce some basic definitions and results of uncertain differential equations. An uncertain differential equation is a type of differential equation driven by a canonical Liu process.

Definition 2.1 (Liu [7]). Suppose C_t is a canonical Liu process, and f and g are some given functions. Then,

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is called an uncertain differential equation. A solution is a Liu process X_t that satisfies the above equation identically in t.

The existence and uniqueness theorem of solution of uncertain differential equation was proved by Chen and Liu [1] under linear growth condition and Lipschitz continuous condition.

Theorem 2.1 (Chen and Liu [1]). *The uncertain differential equation*

$$X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

has a unique solution if the coefficients $f(t, x)$ and $g(t, x)$ satisfy the linear growth condition

$$|f(t,x)| + |g(t,x)| \le L(1+|x|), \quad \forall x \in \Re, \quad t \ge 0$$

and Lipschitz condition

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \le L|x-y|, \quad \forall x, y \in \Re, \quad t \ge 0$$

for some constant L. Moreover, the solution is sample continuous.

More importantly, Yao and Chen [20] proved that the solution of an uncertain differential equation can be represented by a spectrum of ordinary differential equations.

Definition 2.2 (Yao and Chen [20]). Let α be a number with $0 < \alpha < 1$. An uncertain differential equation

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is said to have an α-path X_t^α if it solves the corresponding ordinary differential equation

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t$$

where $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of standard normal uncertain variable, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Theorem 2.2 (Yao-Chen Formula [20]). *Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t.$$

Then

$$\mathcal{M}\{X_t \le X_t^\alpha, \ \forall t\} = \alpha,$$

$$\mathcal{M}\{X_t > X_t^\alpha, \ \forall t\} = 1 - \alpha.$$

Theorem 2.3 (Yao and Chen [20]). *Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t.$$