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On a finite difference scheme for blow up solutions for the Chipot–Weissler equation☆

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ABSTRACT

In this paper, we are interested in the numerical analysis of blow up for the Chipot–Weissler equation $u_t = \Delta u + |u|^{p-1}u - |\nabla u|^q$ with Dirichlet boundary conditions in bounded domain when $p > 1$ and $1 \leq q \leq \frac{2p}{p+1}$. To approximate the blow up solution, we construct a finite difference scheme and we prove that the numerical solution satisfies the same properties of the exact one and blows up in finite time.

1. Introduction

In this paper, we study the numerical approximation of solutions which achieve blow up in finite time of the Chipot–Weissler equation

$$u_t = \Delta u + |u|^{p-1}u - |\nabla u|^q \text{ in } \mathbb{R}^+ \times \Omega, \tag{1}$$

with Dirichlet boundary conditions

$$u(t,x) = 0,\ t > 0 \quad \text{and} \quad x \in \partial\Omega, \tag{2}$$

and initial data

$$u(0,x) = u_0(x), \quad x \in \Omega, \tag{3}$$

where Ω is a regular bounded domain in $\mathbb{R}^d$ and $p,\ q$ are fixed finite parameters.

This problem represents a model in population dynamics which is proposed by Souplet in [18], where (1)–(3) describe the evolution of the population density of a biological species, under the effect of certain natural mechanisms, $u(t, x)$ denotes the spatial density of individuals located near a point $x \in \Omega$ at a time $t \geq 0$. The evolution of this density depends on three types of mechanisms: displacements, births and deaths. The reaction term represents the rate of births. If we suppose that the individuals can be destroyed by some predators during their displacements, then the dissipative gradient term represents the density of predators.

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In this paper, we are concerned with solution u of (1)–(3) which blows up in the L^∞ norm in the following sense: there exists $T^* < \infty$, called the blow up time such that the solution u exists in $[0, T^*)$ and

$$\lim_{t\to T^*} \|u(t)\|_{L^\infty} = +\infty.$$

Numerous articles have been published concerning the problem of global existence or nonexistence of solutions to nonlinear parabolic equations. Problem (1)–(3), has been widely analyzed from a mathematical point of view, on the profile, blow up rates, asymptotic behaviors and self similar solutions (see for example: [15,17,20,22–24]), but to our knowledge, there are no results concerning its numerical approximation.

Let us first have a look at the theoretical analysis of this problem. The quasilinear parabolic Eq. (1) was introduced in 1989 by Chipot and Weissler [5] in order to investigate the effect of a damping term on global existence or nonexistence of solutions. They proved local existence, uniqueness and regularity for the problem and they showed the following theorem:

Theorem 1.1 ([5]). *Suppose $s \ge q,\ s > d(q-1),\ s \ge \frac{dp}{d+p},\ s > \frac{d(p-1)}{p+1},\ s \ge 2q$ and $s > dq$.*

Let $u_0 \in D := \{\phi \in W^{3,s}(\Omega) \cap W_0^{1,s}(\Omega)$ such that $\Delta\phi + |\phi|^{p-1}\phi - |\nabla\phi|^q \in W_0^{1,s}(\Omega)\}$ and let $u(t)$ be the maximal solution of the integral equation associated to (1)–(3)

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}(|u|^{p-1}u - |\nabla u|^q)(s)ds \tag{4}$$

where $e^{t\Delta}$ denotes the heat semigroup on Ω with Dirichlet conditions. Then we have

1. *$u \in C^1([0, T^*); W_0^{1,s}(\Omega))$ and $u'(t) = \Delta u + |u|^{p-1}u - |\nabla u|^q$ where each term on the right side is in $C([0, T^*); L^{\frac{s}{q}}(\Omega))$.*
2. *$u \in C((0, T^*); W^{2,\frac{s}{q}}(\Omega))$.*
3. *$\|u(t)\|_\infty$ and $\|\nabla u(t)\|_\infty$ are bounded on any interval $[0, T]$ with $T < T^*$.*

Remark 1.2. D is the domain of the generator B of the semi-flow W_t on $W_0^{1,s}(\Omega)$ resulting from the integral Eq. (4) (for more details see [5]).

They proved that under appropriate conditions on q, p and d, there exists a suitable initial value u_0 so that the corresponding solution of (1)–(3) blows up in a finite time. More precisely

Theorem 1.3 ([5]). *Let $p > 1,\ \ 1 \le q \le \frac{2p}{p+1}$ and $u_0 \in W^{3,s}(\Omega)$ for s satisfying the same conditions as in Theorem 1.1, u_0 not identically zero. In addition, we suppose that:*

1. *$u_0 = 0$ on $\partial\Omega$.*
2. *$\Delta u_0 - |\nabla u_0|^q + |u_0|^p = 0$ on $\partial\Omega$.*
3. *$u_0 \ge 0$ in Ω.*
4. *$\Delta u_0 - |\nabla u_0|^q + u_0^p \ge 0$ in Ω.*
5. *$E(u_0) = \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{p+1}\|u_0\|_{p+1}^{p+1} \le 0$.*
6. *If $q < \frac{2p}{p+1}$ then $\|u_0\|_{p+1}$ is sufficiently large.*
7. *If $q = \frac{2p}{p+1}$ then p is sufficiently large.*

Then, the corresponding solution of (1)–(3) blows up in finite time in the L^∞ norm.

The obvious difficulty with this result is that it is not at all clear if such a u_0 exists.

Souplet and Weissler have proved in [22] that in a possibly unbounded regular domain Ω, finite time blow up occurs in $W_0^{1,s}$ norm (s sufficiently large), for large data whenever $p > q$, and they give an estimation for the blow up time T^*. More precisely, we have :

Theorem 1.4 ([22]). *Assume $p > q$ and let $\psi \in W_0^{1,s}(\Omega)$ (s sufficiently large) with $\psi \ge 0,\ (\psi \not\equiv 0)$.*

1. *There exists some $\lambda_0 = \lambda_0(\psi) > 0$ such that for all $\lambda > \lambda_0$, the solution of (1)–(3) with initial data $\phi = \lambda\psi$ blows up in finite time in $W_0^{1,s}$ norm. If either Ω is bounded or $q < 2$, then blow up occurs in L^∞ norm.*
2. *If Ω is bounded, there is some $C(\psi) > 0$ such that*

$$\frac{1}{(p-1)(\lambda|\psi|_\infty)^{p-1}} \le T^*(\lambda\psi) \le \frac{C(\psi)}{(\lambda|\psi|_\infty)^{p-1}}, \quad \lambda \to \infty.$$

For more details about the result of regularity of the solution and the conditions of blow up, see [5,12,16,21,22] and the references therein.

In [2,19,23], authors have proved the following estimations about the blow-up rate:

Theorem 1.5. *Assume $q < \frac{2p}{p+1},\ \ p \le 1 + \frac{2}{d+1}$ and let $u \ge 0$ be a solution of (1)–(3) such that $T^* < +\infty$. Then, we have*

$$C_1(T^* - t)^{-\frac{1}{p-1}} \le \|u(t)\|_\infty \le C_2(T^* - t)^{-\frac{1}{p-1}} \quad \text{as} \ \ t \longrightarrow T^*.$$