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Numerical methods of solutions of boundary value problems for the multi-term variable-distributed order diffusion equation

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ABSTRACT

Solutions of the Dirichlet and Robin boundary value problems for the multi-term variable-distributed order diffusion equation are studied. A priori estimates for the corresponding differential and difference problems are obtained by using the method of the energy inequalities. The stability and convergence of the difference schemes follow from a priory estimates. The credibility of the obtained results is verified by performing numerical calculations for test problems.

1. Introduction

Differential equations with fractional order derivatives provide a powerful mathematical tool for accurate and realistic description of physical and chemical processes proceeding in media with fractal geometry [1–5]. It is known that the order of a fractional derivative depends on the fractal dimension of medium [6,7]. It is therefore reasonable to construct mathematical models based on partial differential equations with the variable and distributed order derivatives [1,8–14]. Analytical methods for solving such equations are scarcely effective, so that the development of the corresponding numerical methods is very important.

The initial-boundary-value problems for the generalized multi-term time fractional diffusion equation over an open bounded domain $G \times (0, T)$, $G \in \mathbb{R}^n$ were considered [15]. Multi-term linear and non-linear diffusion-wave equations of fractional order were solved in [16] using the Adomian decomposition method. Applications of the homotopy analysis and new modified homotopy perturbation methods to solutions of multi-term linear and nonlinear diffusion-wave equations of fractional order are discussed in [17,18].

The variable-order nonlinear fractional diffusion equation with a generalized Riesz fractional derivative of variable order was analyzed by applying a new explicit finite-difference approximation [19]. The convergence and stability of this approximation were proved. Explicit and implicit Euler approximations for a variable-order fractional advection-diffusion equation with a nonlinear source term on a finite domain were proposed [20]. The stability and convergence of the methods were discussed. A numerical scheme with first order temporal accuracy and fourth order spatial accuracy for a variable-order anomalous subdiffusion equation was constructed in [21]. The technique of Fourier analysis was used to study the convergence, stability, and solvability of this scheme. Two numerical methods (the implicit and explicit ones) were developed to solve a two-dimensional variable-order anomalous subdiffusion equation [22]. Their stability, convergence and solvability were also investigated by the Fourier method. Analytical solutions for the multi-term time-fractional diffusion-wave and the multi-term time-space Caputo–Riesz fractional advection-diffusion equations on a finite domain are studied in [23,24]. An approximate scheme for the variable order

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time fractional diffusion equation with the Coimbra variable order time fractional operator was introduced [25]. The authors of [26] considered a new space-time variable fractional order advection-dispersion equation on a finite domain. The equation is obtained from the standard advection-dispersion equation by replacing the first-order time derivative by the Coimbra variable fractional derivative of order $\alpha(x) \in (0, 1]$, and the first-order and second-order space derivatives by the Riemann–Liouville derivatives of order $\gamma(x, t) \in (0, 1]$ and $\beta(x, t) \in (1, 2]$, respectively. The time distributed-order and Riesz space fractional diffusion equations on bounded domains with Dirichlet boundary conditions were considered in [27]. The fundamental solution of the multi-term diffusion equation with the Dzharbashyan–Nersesyan fractional differentiation operator with respect to the time variables is constructed in [28]. A priory estimates for the difference problems obtained in [29–31] by using the maximum principle imply the stability and convergence of the considered difference schemes. Using the energy inequality method, a priori estimates for the solution of the Dirichlet and Robin boundary value problems for the fractional and variable order diffusion equation with Caputo fractional derivative have been obtained [32,33].

The present paper develops the method of the energy inequalities to solve the differential as well as difference problems for the subdiffusion equation with a time-fractional derivative of non-trivial structure. The proposed method allows one to find a priori estimates for solutions of a wide class of boundary value problems for the multi-term variable-distributed order diffusion equation to which the maximum principle is not applicable.

2. Boundary value problems in differential setting

2.1. The Dirichlet boundary value problem

In rectangle $\bar{Q}_T = \{(x, t) : 0 \le x \le l, 0 \le t \le T\}$ let us study the boundary value problem

$$\mathbb{P}^{(\theta)}_{(\omega)}(\partial_{0t})u(x,t) = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right) - q(x,t)u + f(x,t), \;\; 0 < x < l, \;\; 0 < t \le T, \tag{1}$$

$$u(0,t) = 0, \quad u(l,t) = 0, \quad 0 \le t \le T, \tag{2}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le l, \tag{3}$$

where

$$\mathbb{P}^{(\theta)}_{(\omega)}(\partial_{0t})u(x,t) = \int_\alpha^\beta d\gamma \sum_{r=1}^m \omega_r(x,\gamma)\partial_{0t}^{\theta_r(x,\gamma)}u(x,t),$$

$$\alpha < \beta, \quad 0 < \theta_r(x,\gamma) < 1, \quad \omega_r(x,\gamma) \ge 0, \quad r = 1, 2, \ldots, m, \quad \text{for all}$$

$$(x,\gamma) \in [0,l] \times [\alpha,\beta], \quad \int_\alpha^\beta d\gamma \sum_{r=1}^m \omega_r(x,\gamma) > 0, \quad \theta_r(x,\gamma) \in C[0,l] \times [\alpha,\beta],$$

$$0 < c_1 \le k(x,t) \le c_2, \quad q(x,t) \ge 0,$$

$\partial_{0t}^{\theta_r(x,\gamma)}u(x,t) = \int_0^t u_\eta(x,\eta)(t-\eta)^{-\theta_r(x,\gamma)}d\eta/\Gamma(1-\theta_r(x,\gamma))$ is a Caputo fractional derivative of order $\theta_r(x,\gamma)$ [34,35].

The existence of the solution for the initial boundary value problem of fractional, multi-term and distributed order diffusion equation has been proven in [12,23,36–39].

Let us assume further the existence of a solution $u(x,t) \in C^{2,1}(\bar{Q}_T)$ for the problems (1)–(3), where $C^{m,n}$ is the class of functions, continuous together with their partial derivatives of the order m with respect to x and order n with respect to t on $\bar{Q}_T$.

Lemma 1. *For any functions $v(t)$ and $w(t)$ absolutely continuous on $[0, T]$, one has the equality:*

$$v(t)\mathbb{P}^{(\bar\theta)}_{(\bar\omega)}(\partial_{0t})w(t) + w(t)\mathbb{P}^{(\bar\theta)}_{(\bar\omega)}(\partial_{0t})v(t) = \mathbb{P}^{(\bar\theta)}_{(\bar\omega)}(\partial_{0t})(v(t)w(t))$$
$$+ \int_\alpha^\beta d\gamma \sum_{r=1}^m \frac{\bar\omega_r(\gamma)\bar\theta_r(\gamma)}{\Gamma(1-\bar\theta_r(\gamma))}\int_0^t \frac{d\xi}{(t-\xi)^{1-\bar\theta_r(\gamma)}}\int_0^\xi \frac{v'(\eta)d\eta}{(t-\eta)^{\bar\theta_r(\gamma)}}\int_0^\xi \frac{w'(s)ds}{(t-s)^{\bar\theta_r(\gamma)}}, \tag{4}$$

where $\bar\omega_r(\gamma) \ge 0$, $0 < \bar\theta_r(\gamma) < 1$, for all $\gamma \in [\alpha, \beta]$, $\int_\alpha^\beta d\gamma \sum_{r=1}^m \bar\omega_r(\gamma) > 0$.

Proof. For any fixed $\gamma \in [\alpha, \beta]$ and $r \in \{1, 2, \ldots, m\}$, relying on Lemma 1 [33] one finds the following equality

$$v(t)\partial_{0t}^{\bar\theta_r(\gamma)}w(t) + w(t)\partial_{0t}^{\bar\theta_r(\gamma)}v(t) = \partial_{0t}^{\bar\theta_r(\gamma)}(v(t)w(t))$$
$$+ \frac{\bar\theta_r(\gamma)}{\Gamma(1-\bar\theta_r(\gamma))}\int_0^t \frac{d\xi}{(t-\xi)^{1-\bar\theta_r(\gamma)}}\int_0^\xi \frac{v'(\eta)d\eta}{(t-\eta)^{\bar\theta_r(\gamma)}}\int_0^\xi \frac{w'(s)ds}{(t-s)^{\bar\theta_r(\gamma)}}. \tag{5}$$

Multiplying (5) by $\bar\omega_r(\gamma)$ and summing the result over r from 1 to m, then integrating over γ from α to β one obtains (4). The proof of Lemma 1 is complete. □