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Nonlinear reaction–diffusion systems with a non-constant diffusivity: Conditional symmetries in no-go case

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ABSTRACT

Q-conditional symmetries (nonclassical symmetries) for a general class of two-component reaction–diffusion systems with non-constant diffusivities are studied. The work is a natural continuation of our paper “Conditional symmetries and exact solutions of nonlinear reaction–diffusion systems with non-constant diffusivities” (Cherniha and Davydovych, 2012) [1] in order to extend the results on so-called no-go case. Using the notion of *Q*-conditional symmetries of the first type, an exhaustive list of reaction–diffusion systems admitting such symmetry is derived. The results obtained are compared with those derived earlier. The symmetries for reducing reaction–diffusion systems to two-dimensional dynamical systems (ODE systems) and finding exact solutions are applied. As result, multiparameter families of exact solutions in the explicit form for nonlinear reaction–diffusion systems with an arbitrary power-law diffusivity are constructed and their properties for possible applicability are established.

1. Introduction

This work is a natural continuation of our paper [1] and is devoted to the investigation of the two-component reaction–diffusion (RD) systems of the form

$$\begin{aligned} U_t &= [D^1(U)U_x]_x + F(U,V), \\ V_t &= [D^2(V)V_x]_x + G(U,V), \end{aligned} \tag{1}$$

where $U = U(t,x)$ and $V = V(t,x)$ are two unknown functions representing, say, the densities of populations (cells, chemicals), $F(U,V)$ and $G(U,V)$ are the given smooth functions describing interaction between them and environment, the functions $D^1(U)$ and $D^2(V)$ are the relevant diffusivities (hereafter they are positive smooth functions) and the subscripts t and x denote differentiation with respect to these variables. The class of RD systems (1) generalizes many well-known nonlinear second-order models and is used to describe various processes in physics, biology, chemistry and ecology. The relevant examples can be found in the well-known books [2–5] and a wide range of papers.

In paper [1], the *Q*-conditional invariance of these systems in the case when the operator in question has the form

$$Q = \xi^0(t,x,u,v)\partial_t + \xi^1(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v,$$

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where $\xi^0 \neq 0$, has been examined and an exhaustive list of reaction–diffusion systems admitting Q-conditional symmetries of the first type [6] has been derived. Here we study so-called no-go case when $\xi^0 = 0$, which is thought to be much more difficult and usually is skipped from examination. The additional reason to avoid examination of no-go case was the well-known fact (firstly proved in [7]) that complete description of Q-conditional symmetries with $\xi^0 = 0$ for scalar evolution equations is equivalent to solving of the equation in question.

On the other hand, the algorithm of heir-equations was proposed in [8], which allows to construct a hierarchy of the conditional symmetry operators starting from a particular one with $\xi^0 = 0$. This algorithm was successfully applied in order to find exact solutions for some evolution equations (in particular, see its application in the recent paper [9]). It can be also noted the very recent paper [10], in which RD systems were investigated in order to find conditional Lie-Bäcklund symmetry (generalized conditional symmetries in terminology of the pioneering paper [11]). However, this is well-known that deriving a complete classification of generalized conditional symmetries is unrealistic even for classes of scalar PDEs because the relevant systems of determining equations are very complicated.

To the best of our knowledge, there are no papers devoted to search Q-conditional symmetry (non-classical symmetry) of the class of systems (1) in the case $\xi^0 = 0$ and application of such symmetries for finding exact solutions of nonlinear reaction–diffusion systems. Here we show that a complete description of Q-conditional symmetries of the first type [6] can be derived in no-go case.

The paper is organized as follows. In section 2, basic definitions are presented, the systems of determining equations are derived and the main theorems are proved. In section 3, the Q-conditional symmetries obtained for reducing of RD systems to systems of ODEs are applied and multiparameter families of exact solutions are constructed. Moreover, it is shown that the solutions obtained possess attractive properties, which may lead to their possible applications. Finally, we summarize and discuss the results obtained in the last section.

2. Conditional symmetry for RD systems

2.1. Definitions and preliminary analysis

Following [1], we simplify RD system (1) by applying the Kirchhoff substitution

$$u = \int D^1(U)dU, \quad v = \int D^1(V)dV, \tag{2}$$

where $u(t, x)$ and $v(t, x)$ are new unknown functions. Hereafter we assume that there exist unique inverse functions to those arising in right-hand-sides of (2). Substituting (2) into (1), one obtains

$$\begin{aligned} u_{xx} &= d^1(u)u_t + C^1(u, v), \\ v_{xx} &= d^2(v)v_t + C^2(u, v), \end{aligned} \tag{3}$$

where the functions d^1, d^2, C^1 and C^2 are uniquely defined via D^1, D^2, F and G by the formulae

$$d^1(u) = \frac{1}{D^1(U)}, \quad d^2(v) = \frac{1}{D^2(V)}, \quad C^1(u, v) = -F(U, V), \quad C^2(u, v) = -G(U, V), \tag{4}$$

where $U = D^1_*(u) \equiv \left(\int D^1(u)du\right)^{-1}, V = D^2_*(v) \equiv \left(\int D^2(v)dv\right)^{-1}$ (the upper subscripts -1 mean inverse functions).

Hereafter we examine class of RD systems (3) instead of (1) because both classes are equivalent. In fact, having any conditional symmetry operator of a RD system of the form (3), one may easily transform those into the relevant operator and a RD system from the class (1) provided the inverse functions in (4) are known.

It is well-known that to find Lie invariance operators, one needs to consider system (3) as the manifold $\mathcal{M} = \{S_1 = 0, S_2 = 0\}$ where

$$\begin{aligned} S_1 &\equiv u_{xx} - d^1(u)u_t - C^1(u, v), \\ S_2 &\equiv v_{xx} - d^2(v)v_t - C^2(u, v), \end{aligned}$$

in the prolonged space of the variables: $t, x, u, v, u_t, v_t, u_x, v_x, u_{xx}, v_{xx}, u_{xt}, v_{xt}, u_{tt}, v_{tt}$. According to the definition, system (3) is invariant under the transformations generated by the infinitesimal operator

$$Q = \xi^0(t, x, u, v)\partial_t + \xi^1(t, x, u, v)\partial_x + \eta^1(t, x, u, v)\partial_u + \eta^2(t, x, u, v)\partial_v, \tag{5}$$

if the following invariance conditions are satisfied:

$$\underset{2}{Q}(S_1)\Big|_{\mathcal{M}} = 0,$$

$$\underset{2}{Q}(S_2)\Big|_{\mathcal{M}} = 0.$$