Applied Mathematics and Computation 268 (2015) 83–92

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On Laplacian energy in terms of graph invariants

Kinkar Ch. Das[a], Seyed Ahmad Mojallal[a], Ivan Gutman[b,c,*]

[a] Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea
[b] Faculty of Science, University of Kragujevac, 34000 Kragujevac, P.O. Box 60, Serbia
[c] State University Novi Pazar, Novi Pazar, Serbia

ARTICLE INFO

Keywords:
Laplacian eigenvalues
Laplacian energy
Vertex connectivity
Edge connectivity
Vertex cover number
Spanning tree packing number

ABSTRACT

For G being a graph with n vertices and m edges, and with Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$, the Laplacian energy is defined as $LE = \sum_{i=1}^{n} |\mu_i - 2m/n|$. Let σ be the largest positive integer such that $\mu_\sigma \geq 2m/n$. We characterize the graphs satisfying $\sigma = n - 1$. Using this, we obtain lower bounds for LE in terms of n, m, and the first Zagreb index. In addition, we present some upper bounds for LE in terms of graph invariants such as n, m, maximum degree, vertex cover number, and spanning tree packing number.

1. Introduction

Let $G = (V, E)$ be a graph of order n with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, $|E(G)| = m$. Let d_i be the degree of the vertex v_i for $i = 1, 2, \ldots, n$. The maximum and the minimum vertex degrees are denoted by Δ and δ, respectively. The so-called first Zagreb index is defined as [9,15,17,34]

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 = \sum_{v_i v_j \in E(G)} (d_i + d_j).$$

Let $\mathbf{A}(G)$ be the $(0, 1)$-adjacency matrix of G and $\mathbf{D}(G)$ the diagonal matrix of vertex degrees. The Laplacian matrix of G is the $n \times n$ matrix $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$. This matrix has nonnegative eigenvalues $n \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. When more than one graph is under consideration, then we write $\mu_i(G)$ instead of μ_i. In what follows, the Laplacian spectrum of the graph G, i.e., the multiset $\{\mu_1(G), \mu_2(G), \ldots, \mu_n(G)\}$ will be denoted by $\mathcal{S}(G)$.

The Laplacian energy of the graph G is defined as [18]

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|. \tag{1}$$

For its basic mathematical properties, including various lower and upper bounds, see [18,30,31,36,37] and especially the most recent works [4–7,12–14,19,35]. It is worth noting that LE found applications not only in theoretical organic chemistry (see, e.g., [16,24,29]), but also in image processing [32] and information theory [21].

At this point it is worth mentioning that in addition to the Laplacian energy, a number of other "graph energies" have been considered. The oldest and most thoroughly studied is the energy based on the eigenvalues of the $(0, 1)$-adjacency matrix [22,23]. Some other are the incidence energy, Randić energy, matching energy, and distance energy, (see, [2,8,20,33] respectively and the references cited therein).

* Corresponding author.
E-mail addresses: kinkardas2003@googlemail.com, ahmad_mojalal@yahoo.com (K.Ch. Das), gutman@kg.ac.rs (I. Gutman).

http://dx.doi.org/10.1016/j.amc.2015.06.064

Let σ $(1 \le \sigma \le n)$ be the largest positive integer such that

$$\mu_\sigma \ge \frac{2m}{n}. \tag{2}$$

Then from Eq. (1) it follows

$$LE(G) = 2\mathcal{S}_\sigma(G) - \frac{4m\sigma}{n} \tag{3}$$

where

$$\mathcal{S}_k(G) = \sum_{i=1}^{k} \mu_i(G).$$

As usual, K_n and $K_{1,\,n-1}$ denote, respectively, the complete graph and the star on n vertices. Further, $K_{n_1,\,n_2,\ldots,\,n_k}$ is the complete k-partite graph whose partition sets are of size $n_1,\ n_2,\ldots,\ n_k$, respectively.

The vertex connectivity $\kappa = \kappa(G)$ of a connected graph G is the minimum number of vertices whose deletion will disconnect G or leave a single vertex. The edge connectivity $\kappa' = \kappa'(G)$ of a connected graph G is the minimum number of edges whose deletion will disconnect G.

The present paper is organized as follows. In Section 2, we list some necessary lemmas and known results. In Section 3, we characterize the graphs satisfying the condition $\sigma = n - 1$. Using this we then obtain a lower bound on $LE(G)$. In Section 4, we present some upper bounds on Laplacian energy.

2. Preliminaries

In this section, we list some previously known results that will be needed in the next two sections.

Lemma 1 ([10]). *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two real symmetric matrices of size n. Then for any* $1 \le k \le n$,

$$\sum_{i=1}^{k} \lambda_i(\mathbf{A}+\mathbf{B}) \le \sum_{i=1}^{k} \lambda_i(\mathbf{A}) + \sum_{i=1}^{k} \lambda_i(\mathbf{B})$$

where $\lambda_i(\mathbf{M})$ *denotes the ith largest eigenvalue of the matrix* $\mathbf{M}$.

Lemma 2 ([25]). *Let G be a simple graph with n vertices. Then the non-increasing Laplacian eigenvalues of* $\mathbf{L}(\overline{G})$ *are* $n - \mu_{n-i}$, $i = 1, 2, \ldots, n-1$, *and 0.*

Lemma 3 ([25]). *Let G be a graph of order n with at least one edge and maximum degree* Δ. *Then*

$$\mu_1(G) \ge \Delta + 1.$$

Moreover, if G is connected, then the equality holds if and only if $\Delta = n - 1$.

Lemma 4 ([11]). *Let T be a tree of order n. Then the sum of the k largest Laplacian eigenvalues of T satisfies*

$$\mathcal{S}_k(T) = \sum_{i=1}^{k} \mu_i(T) \le n + 2k - 2 - \frac{2k-2}{n}.$$

Moreover, the equality is achieved only when $k = 1$ *and* $T \cong K_{1,n-1}$.

The conjugate of a degree sequence (d) is the sequence $(d^*) = (d_1^*,\ d_2^*,\ \ldots,\ d_k^*)$, where $d_i^* = |\{j :\ d_j \ge i\}|$.

Lemma 5 ([3]). *Let G be a connected graph of order n with m edges. In addition, let* d_i^* *be the ith conjugate degree of G. Then*

$$LE(G) \le \sum_{i=1}^{n} \left| d_i^* - \frac{2m}{n} \right|$$

with equality holding if and only if G is a threshold graph.

The union of two vertex disjoint graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph whose vertex (respectively, edge) set is the union of vertex (respectively, edge) sets of G_1 and G_2. The join $G_1 \vee G_2$ of two vertex disjoint graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining each vertex in G_1 with every vertex in G_2. For any positive integers r, s, consider the vertex disjoint graphs G_1, G_2, H_1, H_2 such that G_1, G_2 are complete graphs on r vertices and H_1, H_2 are graphs on s vertices with no edges. In [26], the authors defined the graph $\mathcal{G}(r, s)$ obtained by joining any vertex in subgraph H_1 of $G_1 \vee H_1$ to any vertex in subgraph H_2 of $G_2 \vee H_2$.

Lemma 6 ([26]). *Let G be a graph on* $n \ge 5$ *vertices whose distinct Laplacian eigenvalues are* $0 < \alpha < \beta < \gamma$. *Then the multiplicity of* β *is* $n - 3$ *if and only if G is one of the graphs* $K_1 \vee 2K_{(n-1)/2}$, $\overline{K}_{n/3} \vee 2K_{n/3}$, $K_{n-1} + e$ *or* $\mathcal{G}(r, n/2 - r)$ *for some r.*