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Basins of attraction for several third order methods to find multiple roots of nonlinear equations

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ABSTRACT

There are several third order methods for solving a nonlinear algebraic equation having roots of a given multiplicity m. Here we compare a recent family of methods of order three to Euler–Cauchy's method which is found to be the best in the previous work. There are fewer fourth order methods for multiple roots but we will not include them here.

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1. Introduction

There is a vast literature on the solution of nonlinear equations, see for example Ostrowski [1], Traub [2], Neta [3] and Petković et al. [4]. Here we are interested in algorithms for finding a multiple root of a nonlinear equation $f(x) = 0$. A root α of $f(x)$ is of multiplicity $m > 1$ if $f(\alpha) = 0,\ f^{(i)}(\alpha) = 0$ for $i = 1, 2, \dots, m-1$ and $f^{(m)}(\alpha) \neq 0$. The first method is due to Schröder [5] and it is also referred to as modified Newton,

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}. \tag{1}$$

The method is based on Newton's method for the function $G(x) = \sqrt[m]{f(x)}$ which obviously has a simple root at α, the multiple root with multiplicity m of $f(x)$.

Another method based on the same G is Laguerre's method

$$x_{n+1} = x_n - \frac{\lambda \frac{f(x_n)}{f'(x_n)}}{1 + sgn(\lambda - m)\sqrt{\left(\frac{\lambda - m}{m}\right)\left[(\lambda - 1) - \lambda \frac{f(x_n)f''(x_n)}{f'(x_n)^2}\right]}} \tag{2}$$

where $\lambda\ (\neq 0, m)$ is a real parameter. When $f(x)$ is a polynomial of degree n, this method with $\lambda = n$ is the ordinary Laguerre method for multiple roots, see Bodewig [6]. This method converges cubically. One special case is Euler–Cauchy for $\lambda = 2m$

$$x_{n+1} = x_n - \frac{2m\frac{f(x_n)}{f'(x_n)}}{1 + \sqrt{(2m-1) - 2m\frac{f(x_n)f''(x_n)}{f'(x_n)^2}}}. \tag{3}$$

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Other special cases include Halley's method [7], Ostrowski's method, and Hansen–Patrick's family [8]. Two other cubically convergent methods are: Euler–Chebyshev [2] and Osada's method [9]. Another variation on Chebyshev's method is given by Neta [10]. Sbibih et al. [11] has recently developed a new family of third order methods for multiple roots. The family depends on a weight function given by

$$\begin{aligned} y_n &= x_n - \mu \frac{f(x_n)}{f'(x_n)}, \\ w_n &= \frac{f(y_n)}{f(x_n)} \\ x_{n+1} &= x_n - \phi(w_n) \frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{4}$$

where the weight function ϕ is a complex function, and μ is a non-zero real or complex number. They have shown that the family is of order three, for $m \geq 2$, and of order four for simple roots, if the function ϕ satisfies the following conditions:

$$\begin{aligned} &\phi(t^m) = m \\ &\phi'(t^m) = \tfrac{1}{t^{m-1}(1-t)^2} \\ &\left| \left(\tfrac{1}{\phi'}\right)'(t^m) \right| < \infty \end{aligned} \tag{5}$$

where $t = 1 - \frac{\mu}{m}$.

They have also demonstrated that the following methods are special cases:

- Dong (two methods) [12]
- Victory and Neta [13]
- Neta [10]
- Chun and Neta [14]
- Homeier [15]
- Geum and Kim [16]
- Kim and Geum [17]

The authors picked four different weight functions and compared these four methods to existing ones by solving four nonlinear equations each having a root with a different multiplicity. The members are:

- SSTZ1

$$\begin{aligned} &\phi(x) = ax + b \\ &a = \tfrac{1}{t^{m-1}(1-t)^2} \\ &b = m - \tfrac{t}{(1-t)^2} \end{aligned} \tag{6}$$

- SSTZ2

$$\begin{aligned} &\phi(x) = \tfrac{a}{b-x} \\ &a = m^2 t^{m-1}(1-t)^2 \\ &b = m t^{m-1}(1-t)^2 + t^m \end{aligned} \tag{7}$$

- SSTZ3

$$\begin{aligned} &\phi(x) = x^2 + ax + b \\ &a = \tfrac{1}{t^{m-1}(1-t)^2} - 2t^m \\ &b = m + t^{2m} - \tfrac{t}{(1-t)^2} \end{aligned} \tag{8}$$

- SSTZ4

$$\begin{aligned} &\phi(x) = \tfrac{x^2+ax+b}{(1-x)^2} \\ &a = -2t^m - 2m(1-t^m) + \tfrac{(1-t^m)^2}{t^{m-1}(1-t)^2} \\ &b = t^{2m} + m(1-t^{2m}) - \tfrac{t(1-t^m)^2}{(1-t)^2} \end{aligned} \tag{9}$$

In the next section we will discuss basins of attraction for these four methods and compare the basins with the best known third order method for multiple roots, namely Euler–Cauchy's method (see [18]).