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On one class of persymmetric matrices generated by boundary value problems for differential equations of fractional order

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ABSTRACT

In this paper, we consider the matrix generated by boundary value problems for differential equations of fractional order. In particular, we show that the eigenvalues of these matrices are simple and real.

1. Introduction

For $f(x) \in L^1(0,1)$, the function

$$J_{0|x}^{\alpha} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt \in L^1(0,1)$$

is called the fractional integral of order $\alpha > 0$ on $[0, x]$, and the function

$$J_{x|1}^{\alpha} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_x^1 (t-x)^{\alpha-1} f(t)\, dt \in L^1(0,1)$$

is called the fractional integral of order $\alpha > 0$ on $[x, 1]$ (see [1]). Here, $\Gamma(\alpha)$ is the Gamma function of Euler. As it is known, (see [1]), the function $g(x) \in L^1(0,1)$ given by

$$f(x) = J_{0|x}^{\alpha} g(x)$$

is called the fractional derivative of the function $f(x) \in L^1(0,1)$ of order $\alpha > 0$ on $[0, x]$. In the sequel, we will write

$$g(x) = D_{0|x}^{\alpha} f(x).$$

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The fractional derivative $D^{\alpha}_{x|1} f(x)$ of order $\alpha > 0$ of $f(x) \in L^1(0, 1)$ on $[x, 1]$ is defined similarly.

Let $\alpha \in \mathbb{R}_+$ and $n = [\alpha]$. The operator $D^{\alpha}_{0|x}$ defined by

$$D^{\alpha}_{0|x} f = D^n J^{n-\alpha}_{0|x} f$$

is called the Riemann–Liouville fractional differential operator of order α whenever the right hand side makes sense.

The aim of our study is operators generated by boundary value problems for the following equation:

$$D^{\sigma}_{0|x} u - [\lambda + q(x)]u = 0, \quad 0 < \sigma < \infty. \tag{0.1}$$

Eq. (0.1) is one of the basic equations for modeling the random walk of a point particle which starts to move at the origin of coordinates in $t = 0$ on a self-similar fractal set $\Omega \subset \mathbb{R}^n$, $n \geq 2$.

In [2], various variants of Eq. (0.1) are considered, in particular, the equation $(\sigma = 1 + \gamma)$

$$D^{\sigma}_{0|x} u + [\lambda + q(x)]u(x) = 0, \tag{0.2}$$

where

$$D^{\sigma}_{0|x} u = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_0^x \frac{u'(t)}{(x-t)^{\gamma}}\, dt, \quad 0 < \gamma < 1. \tag{0.3}$$

Eq. (0.2) was studied, for the first time, in [3] as a model equation of fractional order $1 < \sigma < 2$. In particular, in [3] it was established that Eq. (0.2) with $q(x) = 0$ subject to the Dirichlet boundary conditions

$$u(0) = 0, \qquad u(1) = 0, \tag{0.4}$$

is equivalent to the integral equation

$$\int_0^x (x-t)^{1-\gamma} u(t)\, dt - \int_0^1 x^{1-\gamma} (1-t)^{1-\gamma} u(t)\, dt = \frac{\Gamma(2-\gamma)}{\lambda} u. \tag{0.5}$$

In [2], the fractional oscillatory equation [4]

$$\frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{u''(t)}{(x-t)^{\gamma}}\, dt + [\lambda + q(x)]u(x) = 0 \tag{0.6}$$

subject to the Dirichlet boundary conditions

$$u(0) = 0, \quad u(1) = 0,$$

is considered. As it was shown in [5], Eq. (0.6) is equivalent to the integral equation

$$\int_0^x (x-t)^{1-\gamma} u(t)\, dt - \int_0^x x(1-t)^{1-\gamma} u(t)\, dt = \frac{\Gamma(2-\gamma)}{\lambda} u.$$

It was shown in [1] that the operator generated by the expression (0.2) and boundary value conditions of Sturm–Liouville type has oscillatory properties.

Operators such as

$$A^{[\alpha,\beta]}_{\gamma} u(x) = c_{\alpha} \int_0^x (x-t)^{\frac{1}{\alpha}-1} u(t)\, dt + c_{\beta,\gamma} \int_0^1 x^{\frac{1}{\beta}-1} (1-t)^{\frac{1}{\gamma}-1} u(t)\, dt$$

are studied in [6].

In this paper, the operator

$$A_{\rho}(u) = \frac{1}{\Gamma(\rho^{-1})} \left[\int_0^x (x-t)^{\frac{1}{\rho}-1} u(t) dt - \int_0^1 x^{\frac{1}{\rho}-1} (1-t)^{\frac{1}{\rho}-1} u(t)\, dt \right]$$

$(0 < \rho < 1)$ is studied by the matrix methods. It seems to us that matrices generated by operators such as $A^{[\alpha,\beta]}_{\gamma}$ will play the same important role in fractional analysis as the matrices studied in [7] in the theory of oscillations.

2. Matrices generated by fractional differential equations and their main properties

Let us rewrite the kernel $k(x, t)$ of Eq. (0.5) as

$$k(x,t) = \theta(x,t)(x-t)^{\mu} - x^{\mu}(1-t)^{\mu}$$

where $\mu = \frac{1}{\rho} - 1$, $\theta(x,t) = \begin{cases} 0, & x \leq t, \\ 1, & t \leq x. \end{cases}$

We approximate the kernel $k(x, t)$ by a matrix, using the partition of segment [0, 1]:

$$x_0, \; x_i = \frac{i}{n}, \; x_n = 1, \quad t_0 = 0, \; t_j = \frac{j}{n}, \; t_n = 1, \; i = 0, \ldots, n, \; j = 0, \ldots, n.$$