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Comparison criteria for third order functional dynamic equations with mixed nonlinearities

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ABSTRACT

In this paper, we investigate comparison criteria for third order nonlinear dynamic equations with mixed nonlinearities on time scales. Our results are essentially new. Some applications illustrating the importance of our results are included and these applications solve a problem posed in [2, Remark 3.3].

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1. Introduction

We investigate comparison criteria for the third order nonlinear dynamic equation with mixed nonlinearities on time scales of the form

$$[r_2(t)\phi_{\gamma_2}([r_1(t)\phi_{\gamma_1}(x^{\Delta}(t))]^{\Delta})]^{\Delta} + p(t)\phi_{\gamma_2}([r_1(t)\phi_{\gamma_1}(x^{\Delta}(t))]^{\Delta^{\sigma}}) + f(t,x(t)) = 0, \tag{1.1}$$

where

$$f(t,x(t)) := A(t)\phi_{\gamma}(x(h_1(t))) + B(t)\phi_{\beta}(x(h_2(t))) + \int_a^b q(t,s)\phi_{\alpha(s)}(x(h(t,s)))\Delta\zeta(s), \tag{1.2}$$

on a time scale $\mathbb{T}$ which is unbounded above, where $-\infty < a < b \le \infty$ and $r_i \in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $i = 1,2$, where C_{rd} is the space of right-dense continuous functions; $\phi_{\theta}(u) := |u|^{\theta-1}u$, $\theta > 0$; $\alpha \in C_{rd}([a,b)_{\hat{\mathbb{T}}},\mathbb{R}^+)$ is strictly increasing such that $0 \le \alpha(a) < \lambda < \alpha(b-)$ with $\beta > \gamma := \gamma_1\gamma_2 > \lambda > 0$, where $\hat{\mathbb{T}}$ is a time scale; $\zeta \in C_{rd}([a,b)_{\hat{\mathbb{T}}},\mathbb{R})$ is nondecreasing; $p \in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$; and $A, B \in C_{rd}([t_0,\infty)_{\mathbb{T}},[0,\infty))$ and also $q \in C_{rd}([t_0,\infty)_{\mathbb{T}}\times[a,b]_{\hat{\mathbb{T}}},[0,\infty))$. The functions $h_1, h_2 : \mathbb{T} \to \mathbb{T}$ and $h : \mathbb{T}\times\hat{\mathbb{T}} \to \mathbb{T}$ are rd-continuous functions such that

$$\lim_{t\to\infty} h_1(t) = \lim_{t\to\infty} h_2(t) = \lim_{t\to\infty} h(t,s) = \infty \quad \text{for } s \in \hat{\mathbb{T}}.$$

Here $\int_a^b f(s)\Delta\zeta(s)$ denotes the Riemann–Stieltjes integral of the function f on $[a,b)_{\hat{\mathbb{T}}}$ with respect to ζ. We note that as special cases, the integral term in the equation becomes a finite sum when $\zeta(s)$ is a step function and a Riemann integral when $\zeta(s) = s$. For $\hat{\mathbb{T}} = \mathbb{R}$, $n \in \mathbb{N}$, and $s \in [0,n+1)$, we assume that

$$\zeta(s) = \sum_{j=1}^{n} \chi(s-j) \quad \text{with} \quad \chi(s) = \begin{cases} 1, & s \ge 0 \\ 0, & s < 0; \end{cases}$$

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$\alpha \in C[0, n+1)$ such that $\alpha(j) = \alpha_j,\ j = 1, \dots, n$,

$$\alpha_j < \lambda,\ j = 1, 2, \dots, l, \quad \alpha_j > \lambda, \quad j = l+1, l+2, \dots, n; \tag{1.3}$$

$q(t, j) = q_j(t)$ and $h(t, j) = \bar{h}_j(t)$ for $j = 1, \dots, n$. In this case, mixed nonlinearities $f(t, x(t))$ can be written as

$$f(t, x(t)) = A(t)\phi_\gamma(x(h_1(t))) + B(t)\phi_\beta(x(h_2(t))) + \sum_{j=1}^{n} q_j(t)\phi_{\alpha_j}(x(\bar{h}_j(t))).$$

Note that we can get that all terms are sublinear, or superlinear, or a combination of sublinear and superlinear depending on different choices of α_i. For more details, see [3,23]. Throughout this paper, we let

$$x^{[i]} := r_i\phi_{\gamma_i}([x^{[i-1]}]^\Delta),\ i = 1, 2,\ \text{with } x^{[0]} = x.$$

In this case, Eq. (1.1) becomes

$$[x^{[2]}(t)]^\Delta + p(t)\phi_{\gamma_2}([x^{[1]}(t)]^{\Delta^\sigma}) + f(t, x(t)) = 0, \tag{1.4}$$

where $f(t, x(t))$ is defined by (1.2).

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD dissertation written under the direction of Bernd Aulbach (see [21]). Since then a rapidly expanding body of literature has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. Recall that a time scale $\mathbb{T}$ is a nonempty, closed subset of the reals, and the cases when this time scale is the reals or the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [6]). Not only does the new theory of the so-called “dynamic equations” unify the theories of differential equations and difference equations, but also extends these classical cases to cases “in between”, e.g., to the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0}$ (which has important applications in quantum theory (see [22])) and can be applied on different types of time scales such as $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \mathbb{N}_0^2$ and $\mathbb{T} = \mathbb{H}_n$ (the space of harmonic numbers). For an excellent introduction to the calculus on time scales, see Bohner and Peterson [6] and [7].

Although not all solutions of Eq. (1.4) exist on the whole time scale $\mathbb{T}$ for the asymptotic and oscillation purpose, we are only interested in the solutions that are extendable to ∞. Thus, we use the following definition of solutions.

Definition. By a solution of Eq. (1.4) we mean a nontrivial real–valued function $x \in C^1_{rd}([T_x, \infty)_\mathbb{T}, \mathbb{R})$ for some $T_x \ge t_0$ such that $x^{[1]}, x^{[2]} \in C^1_{rd}([T_x, \infty)_\mathbb{T}, \mathbb{R})$, and $x(t)$ satisfies Eq. (1.4) on $[T_x, \infty)_\mathbb{T}$.

Recently, there has been an increasing interest in studying the oscillatory behavior of all order dynamic equations on time scales, we refer the reader to the papers [1,5,6,8–15,16–20,24–32] and the references contained therein.

The study content on the oscillatory and asymptotic behavior of second order dynamic equations on time scales is very rich. In contrast, the study of oscillation criteria of fourth order dynamic equations is relatively less. To the best of our knowledge, the oscillatory behavior of fourth order nonlinear dynamic equations with nonlinear middle term has not been studied till now. Our aim here is to initiate such a study by establishing some new criteria for the oscillation of Eq. (1.4) and some related equations. Our approach is to reduce the problem in such a way that specific oscillation results for first and second order equations can be adapted for the third order case.

2. Main results

In the following, we denote by $L_\zeta(a, b)_{\hat{\mathbb{T}}}$ the set of Riemann–Stieltjes integrable functions on $[a, b)_{\hat{\mathbb{T}}}$ with respect to ζ. Let $c \in [a, b)_{\hat{\mathbb{T}}}$ such that $\alpha(c) = \lambda$. We further assume that $\alpha^{-1} \in L_\zeta(a, b)_{\hat{\mathbb{T}}}$ such that

$$0 \le \alpha(a) < \lambda < \alpha(b-),\ \int_a^c \Delta\zeta(s) > 0 \text{ and } \int_c^b \Delta\zeta(s) > 0.$$

We start with the following two lemmas which generalize Lemmas 2.1 and 2.2 in [18,28].

Lemma 2.1. *There exists $\eta \in L_\zeta(a, b)_{\hat{\mathbb{T}}}$ such that $\eta(s) > 0$ on $[a, b)_{\hat{\mathbb{T}}}$,*

$$\int_a^b \alpha(s)\eta(s)\Delta\zeta(s) = \lambda \quad \text{and} \quad \int_a^b \eta(s)\Delta\zeta(s) = 1. \tag{2.1}$$

Proof. Let

$$m := \lambda\left(\int_c^b \Delta\zeta(s)\right)^{-1} \int_c^b \alpha^{-1}(s)\Delta\zeta(s);$$

$$n := \lambda\left(\int_a^c \Delta\zeta(s)\right)^{-1} \int_a^c \alpha^{-1}(s)\Delta\zeta(s);$$