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Bell polynomials and modified Bessel functions of half-integral order

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ABSTRACT

Some representation formulas for the modified Bessel functions in terms of Bell polynomials are derived. In particular, the cases of the half-integral order modified Bessel functions of the first and second kind are considered. We also consider the case of the half-integral order modified Bessel functions of the third kind (the so called Henkel's functions).

1. Introduction

In a recent paper Ferrante [1] proved, among others results, a relation between the nth derivative of a composite function and the half-integral order modified Bessel functions of the second kind. Since the nth derivative of a composite function can be expressed in terms of Bell polynomials [2], it is natural to conjecture the possibility to derive a Bell-type representation formula for these half-integral order modified Bessel functions.

Being related to partitions, the Bell polynomials often appear in Combinatorial Analysis [3]. They have been also applied in many different situations, such as the Blissard problem (see [3], p. 46), the representation of Lucas polynomials of the first and second kind [4,5], the construction of recurrence relations for a class of Freud-type polynomials [6], etc. However, in our opinion, the most important of their applications is connected with the possibility to represent, by using such a powerful tool, the symmetric function of a countable set of numbers. As a matter of fact, by using Bell polynomials, it is possible to deduce the relations which generalize the classical algebraic Newton–Girard formulas. Consequently, as it was shown in [7], it is possible to find reduction formulas for the *orthogonal invariants* of a strictly Positive Compact Operator, deriving in a simple way the so called Robert formulas [8].

The above mentioned result by Ferrante gives us the possibility to show another application of Bell polynomials. He starts from the following expression for the nth derivative $\phi_s^{(n)}(x)$ of the function $\phi_s(x) = e^{s\sqrt{x}}$

$$\phi_s^{(n)}(x) = \left(\frac{s}{2\sqrt{x}}\right)^n e^{s\sqrt{x}} \sum_{k=0}^{n-1} \left(\frac{1}{s\sqrt{x}}\right)^k \frac{(-1)^k (n+k-1)!}{2^k k! (n-k-1)!}, \qquad x > 0, \tag{1.1}$$

where s is a real, unessential, parameter and $n = 1, 2, \ldots$.

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Eq. (1.1) shows a relation of this nth derivative with the half-integral order modified Bessel functions of the second kind, defined as (see [9], p. 925)

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2z)^k}, \qquad z \in \mathbb{C}\,. \tag{1.2}$$

By putting $s = 1$ in Eq. (1.1), and therefore considering the nth derivative $\varphi^{(n)}(x)$ of the composite function $\varphi(x) = e^{\sqrt{x}}$, the result by Ferrante writes:

$$K_{n-\frac{1}{2}}(-\sqrt{x}) = \sqrt{\pi}\,(-1)^{-1/2}\,(2\sqrt{x})^{n-1/2}\,\varphi^{(n)}(x), \qquad x > 0, \tag{1.3}$$

for $n = 1, 2, \ldots$.

Some other important relation formulas for the nth derivative $\phi_s^{(n)}(x)$ and the half-integral order modified Bessel functions are listed in [10, p. 5], in two entries, 1.1.3.7 and 1.1.3.11.

In this article, after recalling in Section 2 the Bell polynomials, we will first give some relations, similar to Eq. (1.3), involving the half-integral order modified Bessel functions of the first kind,

$$I_{\pm(n+\frac{1}{2})}(z) = I^{(1)}_{n+\frac{1}{2}}(z) \pm I^{(2)}_{n+\frac{1}{2}}(z), \qquad z \in \mathbb{C}, \tag{1.4}$$

where

$$I^{(1)}_{n+\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}}\, e^{z} \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!\,(n-k)!\,(2z)^k}, \tag{1.5}$$

$$I^{(2)}_{n+\frac{1}{2}}(z) = \frac{(-1)^{n+1}}{\sqrt{2\pi z}}\, e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2z)^k} \tag{1.6}$$

and the half-integral order modified Bessel functions of third kind (or Henkel's functions)

$$H^{(1)}_{n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\, i^{-n}\, e^{iz} \sum_{k=0}^{n-1} (-1)^k \frac{(n+k-1)!}{k!\,(n-k-1)!\,(2iz)^k}, \qquad z \in \mathbb{C} \tag{1.7}$$

$$H^{(2)}_{n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\, i^{n}\, e^{-iz} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{k!\,(n-k-1)!\,(2iz)^k}, \qquad z \in \mathbb{C}. \tag{1.8}$$

Furthermore, in the last section, we will give some explicit representations of all these half-integral order modified Bessel functions in terms of the Bell polynomials, since these polynomials are considered as the standard mathematical tool for representing the nth derivative of a composite function.

2. Recalling the Bell polynomials

The problem of finding an explicit expression for the nth derivative of a composite function was first solved by Faà di Bruno [11]. The relevant problem of finding an efficient computational method was solved by Bell, by means of the introduction of his polynomials [2], which can be computed recursively, whereas the Faà di Bruno formula is based on the partitions of the integer n, a set whose cardinality increases in extremely fast way.

Consider the composite function $\Phi(x) := f(g(x))$ of functions $t = g(x)$ and $y = f(t)$ defined in suitable intervals of the real axis and n times differentiable with respect to the relevant independent variables. By using the following notations:

$$\Phi_m := D_x^m \Phi(x), \qquad f_h := D_t^h f(t)|_{t=g(x)}, \qquad g_k := D_x^k g(x),$$

the nth derivative can be represented by

$$\Phi_n := Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n), \tag{2.1}$$

where the Y_n are, by definition, the Bell polynomials.

For example one has:

$$\begin{aligned} Y_1(f_1, g_1) &= f_1 g_1 \\ Y_2(f_1, g_1; f_2, g_2) &= f_1 g_2 + f_2 g_1^2 \\ Y_3(f_1, g_1; f_2, g_2; f_3, g_3) &= f_1 g_3 + f_2(3 g_2 g_1) + f_3 g_1^3. \end{aligned}$$

Further examples can be found in [3], p. 49.