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On the first integral and equivalence of nonlinear differential equations☆

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ABSTRACT

In this paper, we study the equivalence of several different types of differential systems and provide a new method to find out the first integral of such differential systems. We draw some sufficient conditions for a general time-varying nonlinear differential system to be equivalent to a given differential system. We use the obtained results to discuss the qualitative behavior of the periodic solutions of such time-varying differential systems and get some interesting conclusions.

1. Introduction

Since Mironenko in literature [1] established the theory of the reflecting function, many scholars are interested in applications of the reflecting theory to study the qualitative behavior of the differential systems and obtain many interesting results [1–16]. Mironenko [1–5] combined the theory of reflecting function with the integral manifolds theory to discuss the symmetry and other geometric properties of the solutions of some differential systems. Alisevich [6] discussed the case when a linear system has triangular reflecting function. Musafirov [7] studied the case when a linear system has reflecting function which can be expressed as a product of three exponential matrices. Maiorovskaya [9] established the sufficient conditions under which the quadratic systems have linear reflecting function. Zhou [14,17] discussed the structure of the reflecting function of some polynomial differential systems, and applied the obtained conclusions to study the qualitative behavior of solutions of such systems.

In the present section, we shall briefly introduce the concept of the reflecting function, which will be used throughout the rest of this article.

Consider differential system

$$x' = X(t, x), \quad t \in \mathbb{R}, x \in D \subset \mathbb{R}^n, \tag{1}$$

which has a continuously differentiable right-hand side and general solution $\varphi(t; t_0, x_0)$. For such a system, the **reflecting function** is defined as $F(t, x) := \varphi(-t, t, x)$ [1].

If system (1) is 2ω-periodic with respect to t, then $T(x) := F(-\omega, x)$ is the Poincaré mapping of (1) over the period $[-\omega, \omega]$. Thus, the solution $x = \varphi(t; -\omega, x_0)$ of (1) defined on $[-\omega, \omega]$ is 2ω-periodic if and only if x_0 is a fixed point of $T(x)$, and the character of stability of this periodic solution is the same as this of the fixed point.

If the reflecting functions of two differential systems coincide in their common domain, then these systems are said to be **equivalent**.

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If $F(t, x)$ is the reflecting function of (1), then it is also the reflecting function of any system

$$x' = X(t,x) + F_x^{-1}R(t,x) - R(-t, F(t,x)), \tag{2}$$

where $R(t, x)$ is an arbitrary vector function such that the solutions of every system above are uniquely determined by its initial conditions. That is to say, all the systems in the form of (2) are equivalent. If these equivalent systems are 2ω-periodic with respect to t, then their Poincaré mappings coincide, the initial conditions at $t = -\omega$ of their $2\omega-$ periodic solutions and their stability characters are the same. Thus, to study the equivalence of two differential systems is very important and interesting.

Unfortunately, in general, it is very difficult to find out the reflecting function of (1), so, to write out the system (2) is difficult, too. How to judge two systems are equivalent when we do not know their reflecting function? This is a very interesting problem! Mironenko in [4,5] has studied it and obtained some excellent results.

Lemma 1 ([4]). *If continuously differentiable vector functions $\Delta_i(t, x)$ $(i = 1, 2, \ldots, m)$ are the solutions of the differential system*

$$\Delta_t(t,x) + \Delta_x(t,x)X(t,x) = X_x(t,x)\Delta(t,x), \tag{3}$$

then all the perturbed systems of the form

$$x' = X(t,x) + \sum_{i=1}^{m} \alpha_i(t)\Delta_i(t,x),$$

where the $\alpha_i(t)$ are arbitrary continuous scalar odd functions, are equivalent to each other and to system (1). In addition, if these equivalent systems are $2\omega-$periodic with respect to t, then the initial conditions at $t = -\omega$ of their $2\omega-$ periodic solutions and their stability characters coincide.

Veresovich [8] researched the case when the non-autonomous two-dimensional quadratic systems are equivalent to a linear system. Belsky in papers [10–13] discussed respectively when a first-order polynomial differential equation is equivalent to a linear equation $(n = 1)$, or the Ricatti equation $(n = 2)$ or the Abel equation $(n = 3)$. In paper [16], I generalized the results of Belsky and studied the equivalence between two arbitrary polynomial differential equations.

In paper [14], Belsky studied the conditions under which the reflecting function of a given two-dimensional quadratic system coincides with that of a quadratic triangular system of the form

$$\begin{cases} x' = a_0(t) + a_1(t)x + a_2(t)x^2, \\ y' = b_0(t) + b_1(t)x + b_2(t)y + a_3(t)xy. \end{cases}$$

In this paper, we focus on the equivalence of some general quadratic systems and some different types of nonlinear differential systems. We give some sufficient conditions of the equivalence of such systems. At the same time, we also provide a new method to find out the first integral of such differential systems.

2. Main results

Theorem 1. *Suppose that $\Delta(t,\ x)$ is a solution of (3), $H(t,x) = c$ (c is a constant) is the first integral of (1). Then $\delta(t,x) = \phi(H(t,x))\Delta(t,x)$ is a solution of (3), too. Therefore, the system (1) is equivalent to system*

$$x' = X(t,x) + \alpha(t)\phi(H(t,x))\Delta(t,x), \tag{4}$$

where $\alpha(t)$ is an arbitrary continuous scalar odd function, $\phi(u)$ is an arbitrary continuously differentiable scalar function.

Proof. As

$$\Delta_t + \Delta_x X(t,x) = X_x(t,x)\Delta,$$

and

$$H_t + H_x X = 0,$$
$$\delta_t + \delta_x X = \phi'(H)(H_t + H_x X)\Delta + \phi(H)(\Delta_t + \Delta_x X) = \phi(H)X_x\Delta = X_x\delta.$$

Thus, δ is a solution of (3), too. By Lemma 1, the system (1) is equivalent to system (4). □

Corollary 1. *The Hamilton system*

$$\begin{cases} x' = H_y(x,y), \\ y' = -H_x(x,y) \end{cases}$$

is equivalent to system

$$\begin{cases} x' = H_y(x,y) + \alpha(t)\phi(H(x,y))H_y(x,y), \\ y' = -H_x(x,y) - \alpha(t)\phi(H(x,y))H_x(x,y) \end{cases}$$

where the definition of functions $\alpha(t)$ and $\phi(u)$ are the same as in Theorem 1.