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A modified product preconditioner for indefinite and asymmetric generalized saddle-point matrices[☆]

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ABSTRACT

In this paper, we present a new modified product preconditioner (MP) for a class of large sparse linear systems with indefinite and asymmetric matrices. The eigenvalue distribution and form of the eigenvectors of the presented new preconditioned matrix and its minimal polynomial are investigated. Some numerical experiments illustrate that the proposed new preconditioner performs better than block diagonal preconditioner, block triangular preconditioner, constraint preconditioner and product preconditioner.

1. Introduction

In this paper, we consider block 2×2 nonsingular matrices of the form

$$\mathcal{A} \equiv \begin{bmatrix} A & B^T \\ C & -D \end{bmatrix}, \tag{1.1}$$

where

$$A \in \mathcal{R}^{n\times n}, \quad B, C \in \mathcal{R}^{m\times n}, \quad D \in \mathcal{R}^{m\times m} \text{ with } n \geq m.$$

Under suitable partitioning, any $(n+m) \times (n+m)$ matrix can be cast in the form above. This linear system (1.1) can be regarded as a generalized saddle point problem, which arises in a variety of scientific and engineering applications, such as computational fluid dynamics, constrained optimization, optimal control, weighted least-squares problems, electronic networks and computer graphics, and typically result from mixed or hybrid finite element approximation of second-order elliptic problems or the Stokes equations and so on; see [1–6] and the references therein. When $B = C$ and $D = 0$, it leads to a classical saddle point problem. In general, matrices A and B are large and sparse. So, it is more attractive to use iterative methods instead of direct methods to solve the problem (1.1).

In recent years, many authors pay more attention to preconditioned iterative methods [7–10,15–20,34,35] for solving large sparse linear systems in terms of lower requirement for storage and fast convergence. A large amount of work has been devoted to studying preconditioners, such as, block diagonal preconditioner (BD) [12,13], block triangular preconditioner (BT)

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[11,14,22,23,25] , constraint preconditioner (CP) [21,24,26–28] and product preconditioner (PP) [23,29]. And there are also some other papers contributing to indefinite large sparse linear systems [23,31,32].

The linear system (1.1) can be rewritten as the following form

$$\mathcal{A}u = b, \tag{1.2}$$

where

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ C & D \end{bmatrix}, \quad u = \begin{bmatrix} x \\ y \end{bmatrix}, \quad b = \begin{bmatrix} f \\ g \end{bmatrix}.$$

Sturler and Liesen proposed the following block diagonal preconditioner(BD) [12] for solving the linear systems (1.1):

$$\mathcal{M}_{\text{bd}} = \begin{bmatrix} G & 0 \\ 0 & D + CG^{-1}B^T \end{bmatrix}. \tag{1.3}$$

Cao presented a block triangular preconditioner(BT) [11]:

$$\mathcal{M}_{\text{bt}} = \begin{bmatrix} G & B^T \\ 0 & -(D + CG^{-1}B^T) \end{bmatrix}. \tag{1.4}$$

Furthermore, Benzi and Szyld in [30], proposed a constraint preconditioner (SC):

$$\mathcal{M}_{\text{sc}} = \begin{bmatrix} G & B^T \\ C & -D \end{bmatrix}. \tag{1.5}$$

To solve the linear systems (1.1), recently, Wang, Huang and Wen in [29], obtained a product preconditioner (PP):

$$\mathcal{M}_{\text{ps}} = \begin{bmatrix} A & AG^{-1}B^T \\ C & -D \end{bmatrix}, \tag{1.6}$$

which is constructed through the product of the block diagonal preconditioner [12] and the constraint preconditioner. Moreover, authors pointed out that the proposed product preconditioner performs better than block diagonal preconditioner, block triangular preconditioner as well as constraint preconditioner.

Inspired by [29], by modifying the product preconditioner, we construct a new preconditioner for a class of large sparse linear systems with indefinite and asymmetric matrices (1.1) in this paper. The eigenvalue distribution and form of the eigenvectors of the presented new preconditioned matrix and its minimal polynomial are also discussed. Some numerical experiments illustrate that the proposed new preconditioner has better performance than block diagonal preconditioner, block triangular preconditioner, constraint preconditioner and product preconditioner in terms of the number of iterations and the computational time.

The remainder of the paper is organized as follows. In Section 2, a new preconditioner for a class of large sparse linear systems with indefinite and asymmetric matrices (1.1) is proposed. In Section 3, the eigenvalue distribution and form of the eigenvectors of the presented new preconditioned matrix and its minimal polynomial are investigated. Some numerical experiments are given in Section 4. At last, conclusions are given in Section 5.

2. The modified product preconditioner

Recently, Wang, Huang and Wen proposed a product preconditioner, which is defined as a nonsingular matrix of the following form:

$$\mathcal{M}_{\text{ps}} = \begin{bmatrix} A & AG^{-1}B^T \\ C & -D \end{bmatrix}, \tag{2.1}$$

where $G \in \mathcal{R}^{n\times n}$ is an approximation of A, but $G \neq A$.

And the product preconditioned matrix $\mathcal{M}_{\text{ps}}^{-1}\mathcal{A}$ can be expressed as

$$\mathcal{M}_{\text{ps}}^{-1}\mathcal{A} = \begin{bmatrix} I & A^{-1}B^T - G^{-1}B^T(D + CG^{-1}B^T)^{-1}(D + CA^{-1}B^T) \\ 0 & (D + CG^{-1}B^T)^{-1}(D + CA^{-1}B^T) \end{bmatrix}. \tag{2.2}$$

As pointed out in [29], when $G = A$, the eigenvalues of the product preconditioned matrix $\mathcal{M}_{\text{ps}}^{-1}\mathcal{A}$ are all unity, hence, the exact solution of the preconditioned linear system $\mathcal{M}_{\text{ps}}^{-1}\mathcal{A}u = \mathcal{M}_{\text{ps}}^{-1}b$ can be obtained in just one iteration. However, it is almost impossible from the point of view of storage requirements. Motivated by this paper, we add a positive parament to (2.2) and get a new preconditioned matrix $\hat{\mathcal{M}}_{\text{mps}}^{-1}\mathcal{A}$ as follows:

$$\hat{\mathcal{M}}_{\text{mps}}^{-1}\mathcal{A} = \begin{bmatrix} I & \alpha A^{-1}B^T - G^{-1}B^T(D + \alpha CG^{-1}B^T)^{-1}(D + \alpha CA^{-1}B^T) \\ 0 & (D + \alpha CG^{-1}B^T)^{-1}(D + \alpha CA^{-1}B^T) \end{bmatrix}. \tag{2.3}$$

Obviously, when the positive parament $\alpha \to 0$, the eigenvalues of the new preconditioned matrix $\hat{\mathcal{M}}_{\text{mps}}^{-1}\mathcal{A}$ are all quite close to unity whether G equals to A or not.