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On the general trigonometric sums weighted by character sums

Qing Tian*, Wei Yang, Liping Ding

School of Science, Xi'an University of Architecture and Technology, Xi'an, 710055, Shaanxi, PR China

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ABSTRACT

The main purpose of this paper is using the estimates for character sums and the mean value properties of Dirichlet L-function to study the hybrid mean value properties of the general trigonometric sums, which is weighted by the $2k$th and first power mean of character sums in $[1, \frac{p}{4})$. The two sharp asymptotic formulae are derived and some interesting connections between these sums are established.

1. Introduction and main results

Let $q \geq 3$ be an integer, χ be a Dirichlet character modulo q. The estimate for the character sums $\sum_{a=N+1}^{N+H} \chi(a)$ plays an important role in number theory. Over the past several decades, many scholars and experts investigated various of properties of that. For example, Pólya [30] and Vinogradov [33] put forth the famous inequality

$$\left| \sum_{a=1}^{x} \chi(a) \right| \leq c\sqrt{p} \ln p,$$

where $q = p$ is a prime. Burgess [5] proved a conjecture of Norton [29] that

$$\sum_{n=1}^{k} \left| \sum_{m=1}^{h} \chi(n+m) \right|^2 < kh,$$

where h is a positive integer. Then he [6,7] derived an upper bound for the fourth power moments

$$\sum_{\chi \neq \chi_0} \sum_{n=1}^{p} \left| \sum_{m=1}^{h} \chi(n+m) \right|^4 < 6p^2h^2.$$

In order to obtain an asymptotic formula for higher moments of character sums, Xu and Zhang [34] studied the $2k$th power mean value of the even primitive character sums in the short interval $[1, \frac{q}{4})$, and obtained a sharp asymptotic formula as

$$\sideset{}{^*}\sum_{\chi(-1)=1} \left| \sum_{n<\frac{q}{4}} \chi(n) \right|^{2k} = \frac{J(q)q^k}{16} \left(\frac{\pi}{8}\right)^{2k-2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p\nmid 2q} A(0, k, p, 2) + O(q^{k+\epsilon}), \tag{1}$$

* The corresponding author. Tel.: +86 13572283507.
E-mail address: qingtian@xauat.edu.cn (Q. Tian).

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where $\sum^*_{\chi(-1)=1}$ denotes the summation over all primitive characters modulo q with $\chi(-1)=1$, $J(q)$ the number of all primitive characters modulo q, $\prod_{p|q}$ the product over all prime divisors p of q,

$$A(m,k,p,s)=\sum_{i=0}^{2k-2}\frac{1}{p^{is}}\sum_{j=0}^{i}(-1)^j C_{2k-1}^j C_{k+m+i-j-1}^{m+i-j} C_{k+i-j-1}^{i-j}$$

with $C_m^n=\frac{m!}{n!(n-m)!}$, and ϵ any fixed positive number.

Let p be a prime and $f(x)=a_s x^s+a_{s-1}x^{s-1}+\cdots+a_0$ be an s-degree polynomial with integral coefficients and $p\dagger(a_s,a_{s-1},\ldots,a_0)$. Ren and Ma [26] defined the general trigonometric sums as

$$\sum_{a=1}^{p}\chi(a)e\left(\frac{f(a)}{p}\right),$$

which is a generalization of the famous classical trigonometric sums

$$\sum_{a=1}^{p-1}e\left(\frac{f(a)}{p}\right).$$

Mordell [28] obtained the famous upper bound

$$\sum_{x=1}^{p-1}e\left(\frac{f(x)}{p}\right)\ll p^{1-\frac{1}{s}},$$

which was extended by Hua [19] and Min [27] to the case of two variables as

$$\sum_{x=1}^{p-1}\sum_{y=1}^{p-1}e\left(\frac{f(x,y)}{p}\right)\ll p^{2-\frac{2}{s}},$$

where $f(x, y)$ is an $s-$degree polynomial with the two variables x and y. Soon after that, Carlitz and Uchiyama [8] improved the upper bound to

$$\left|\sum_{x=1}^{p-1}e\left(\frac{f(x)}{p}\right)\right|\le s\sqrt{p},$$

where $s\ge 2$.

It is surprising that although the value of the general trigonometric sums vary irregularly, their weighted sums enjoy good mean value properties (see [26,35] and references therein). In this paper, we shall try to reveal this point. That is to say, we shall use the estimates for character sums and the mean value properties of Dirichlet L-function to study the mean value properties of the general trigonometric sums weighted by the 2kth and first power mean of character sums in the short interval $[1,\frac{p}{4})$. Some interesting connections between the character sums and the general trigonometric sums are established in this way.

Before proving the theorems, we have to introduce an important multiplicative function $r(n)$. For any prime p and positive integer α, define

$$r(1)=1,\quad r(p^\alpha)=\frac{1}{4^\alpha}C_{2\alpha}^\alpha.$$

Then for any positive integer n with standard factorization $p_1^{\alpha_2}p_1^{\alpha_2}\ldots p_k^{\alpha_k}$, we can easily get

$$r(n)=\frac{1}{4^{\alpha_1+\alpha_2+\cdots+\alpha_k}}C_{2\alpha_1}^{\alpha_1}C_{2\alpha_2}^{\alpha_2}\cdots C_{2\alpha_k}^{\alpha_k}.$$

Besides its multiplicative property, Ren and Zhang [32] proved the beautiful identity that

$$\sum_{d|n}r(d)r\left(\frac{n}{d}\right)=1,\tag{2}$$

which later will help us deal with the first mean value moment of the character sums.

Theorem 1. *Let $p\ge 5$ be a prime and χ denote the Dirichlet character modulo p. Let $f(x)=\sum_{i=0}^{s}a_ix^i$ be a polynomial with $deg(f(x))=s$ and $p\dagger(a_0,a_1,\ldots,a_s)$. Then for any positive integer k and s, we have the asymptotic formula*

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{f(a)}{p}\right)\right|^2\left|\sum_{n\le\frac{p}{4}}\chi(n)\right|^{2k}=\left(C(k)\zeta^{2k-1}(2)+\frac{\pi^{2k-2}}{2^{6k-2}}\right)p^{k+2}\left(1-\frac{1}{p^2}\right)^{2k-1}\prod_{p_0\neq p}A(0,k,p_0,2)+O(p^{k+2-\frac{1}{s}+\epsilon}),$$