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Explicit iterations and extremal solutions for fractional differential equations with nonlinear integral boundary conditions☆

Lihong Zhang[a], Bashir Ahmad[b], Guotao Wang[a,*]

[a] School of Mathematics and Computer Science, Shanxi Normal University, Linfen, Shanxi 041004, People's Republic of China
[b] Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

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ABSTRACT

This paper studies the existence of extremal solutions for nonlinear fractional differential equations with nonlinear integral boundary conditions and explores an explicit algorithm which converges to the extremal solutions of the problem at hand. An example is discussed for the illustration of the main work.

1. Introduction

Consider the following nonlinear fractional integral boundary value problem:

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, u(t), u(\theta(t))), \ n < \alpha \le n+1, \ n \ge 2, \ n \in N, \ t \in J = [0,1], \\ u'(0) = u''(0) = u'''(0) = \cdots = u^{(n)}(0) = 0, \\ u(0) = \int_0^1 g(s, u(s))ds + \lambda, \end{cases} \tag{1.1}$$

where $t \in J = [0, 1]$, ${}^{C}D^{\alpha}$ is the standard Caputo fractional derivative and $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(J \times \mathbb{R}, \mathbb{R})$, $\theta \in C(J, J)$, $\lambda \ge 0$.

Fractional calculus has been investigated in diverse directions by several researchers. The recent development covers the theoretical as well as potential applications of the subject in physical and technical sciences. Specific examples include physics, chemistry, biomathematics, signal and image processing, viscoelasticity, electrical networks, porous media, aerodynamics, modeling for physical phenomena exhibiting anomalous diffusion, economics, and so forth. An important characteristic of a fractional-order differential operator is its nonlocal nature that takes into account the hereditary properties of many materials and processes. This aspect of fractional-order operators has motivated the modelers to make use of the tools of fractional calculus in the mathematical modelling of many real world problems. For further details, we refer the reader to the texts [1,2].

The study of boundary value problems in the setting of fractional calculus has received a great attention in the last decade and a variety of results concerning the existence of solutions, based on various analytic techniques, can be found in the literature [3–22]. The existence theory for fractional boundary value problems, no doubt, provides the basis for onward exploration of the subject. Once the existence of a solution is established, it is equally important to find it, preferably in an analytic form.

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* Corresponding author. Tel.: +86 18935042188/98.
E-mail addresses: zhanglih149@126.com (L. Zhang), bashirahmad_qau@yahoo.com (B. Ahmad), wgt2512@163.com (G. Wang).

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An interesting and useful analytic strategy that not only ensures the existence of solutions for such problems but also provides means for finding them is the monotone iterative technique coupled with the concept of upper and lower solutions. For examples and details, see [23–32]. Motivated by this approach, we seek the extremal solutions for the problem (1.1).

The paper is organized as follows. In Section 2, we obtain an auxiliary lemma that plays a key role in establishing the proposed work. A comparison result is also discussed. Section 3 contain the main result and an example illustrating it.

2. Preliminaries

Definition 2.1. We say that $u(t)$ is called a lower solution of problem (1.1) if

$$\begin{cases} {}^CD^\alpha u(t) \le f(t, u(t), u(\theta(t))), \ n < \alpha \le n+1, \ n \ge 2, \ n \in N, \\ u'(0) = u''(0) = u'''(0) = \cdots = u^{(n)}(0) = 0, \\ u(0) \le \int_0^1 g(s, u(s))ds + \lambda, \end{cases}$$

and it is an upper solution of (1.1) if the above inequalities are reversed.

Definition 2.2. The Riemann–Liouville fractional integral of order α for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0,$$

provided that such integral exists.

Definition 2.3. For at least n-times absolutely continuously differentiable function $f : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$${}^CD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1. *Let* $n < \alpha \le n+1$, $n \ge 2$, $n \in N$, $\xi \ne 1$ *and* $y \in C[0, 1]$. *Then the linear fractional integral boundary value problem*

$$\begin{cases} {}^CD^\alpha u(t) = y(t), \quad 0 < t < 1, \\ u'(0) = u''(0) = u'''(0) = \cdots = u^{(n)}(0) = 0, \\ u(0) = \xi \int_0^1 u(s)ds + \lambda, \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s)y(s)ds + \frac{\lambda}{1-\xi},$$

where

$$G(t, s) = \begin{cases} \dfrac{\alpha(t-s)^{\alpha-1}(1-\xi) + \xi(1-s)^\alpha}{(1-\xi)\Gamma(\alpha+1)}, & 0 \le s \le t \le 1, \\ \dfrac{\xi(1-s)^\alpha}{(1-\xi)\Gamma(\alpha+1)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.2}$$

Proof. One can transform the equation ${}^CD^\alpha u(t) = y(t)$ to an equivalent integral equation

$$u(t) = I^\alpha y(t) + \sum_{i=0}^n b_i t^i = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \sum_{i=0}^n b_i t^i,$$

for some $b_i \in \mathbb{R} (i = 0, 1, 2, \ldots, n)$.

Applying the conditions $u'(0) = u''(0) = u'''(0) = \cdots = u^{(n)}(0) = 0$ and $u(0) = \xi \int_0^1 u(s)ds + \lambda$, we obtain that $b_1 = b_2 = b_3 = \cdots = b_n = 0$ and

$$b_0 = \xi \int_0^1 u(s)ds + \lambda.$$

So, it holds that

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \xi \int_0^1 u(s)ds + \lambda. \tag{2.3}$$

Letting $\int_0^1 u(s)ds = B$, and integrating both sides of (2.3), we have