Applied Mathematics and Computation 268 (2015) 413–421

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# Hybrid Ikebe–Newton's iteration for inverting general nonsingular Hessenberg matrices

J. Abderramán Marrero [a,*], M. Rachidi [b]

[a] Department of Mathematics Applied to Information Technology, School of Telecommunication Engineering, UPM - Technical University of Madrid, Avda Complutense s/n. Ciudad Universitaria, 28040 Madrid, Spain
[b] Group of DEFA - Department of Mathematics and Informatics, Faculty of Sciences, University Moulay Ismail, B.P. 4010, Beni M'hamed, Meknes, Morocco

ARTICLE INFO

*MSC:*
15A09
15A23
65F99

*Keywords:*
Accuracy
Hessenberg matrix
Matrix inverse
Newton's method

ABSTRACT

After a concise survey, the expanded Ikebe algorithm for inverting the lower half plus the superdiagonal of an $n \times n$ unreduced upper Hessenberg matrix $H$ is extended to general nonsingular upper Hessenberg matrices by computing, in the reduced case, a block diagonal form of the factor matrix $H_L$ in the inverse factorization $H^{-1} = H_L U^{-1}$. This factorization enables us to propose hybrid and accurate (nongaussian) procedures for computing $H^{-1}$. Thus, $H_L$ is computed directly in the aim to be used as a fine initial guess for Newton's iteration, which converges to $H^{-1}$ in a suitable number of iterations.



## 1. Introduction

In addition to their main role in linear algebra and matrix theory, the inverses of nonsingular square matrices have a lot of applications in scientific and engineering problems. It is remarkable its increasing utility in analyzing stimulus-response relationships of natural and artificial systems. The more extended and practical method for inverting square matrices is the *LU* method with partial pivoting strategies, and then using backward (or forward) substitution in the inversion of the resulting triangular matrices [1].

The $n \times n$ nonsingular Hessenberg matrices $H = (h_{ij})_{i,j=1}^{n}$, with $h_{ij} = 0$ for $i \geq j+2$, are prototypical. These matrices are profusely used in control theory as a consequence of its impact in the eigenvalue problem, after reducing a square nonsingular matrix to its Hessenberg form by unitary (Givens or Householder) transformations in $O(n^3)$ time. Without loss of generality, we consider upper Hessenberg matrices. Closed form representations for the entries of the inverses of nonsingular Hessenberg matrices, based on determinants of some principal submatrices, are known [2]. These are of theoretical interest. As a rule, when computing the large recurrence relations of the involved determinants, bad numerical performance is observed [3]. The Hessenberg matrices can be inverted in $O(n^3)$ time using substitution schemes. Indeed, *LU*-factorizable Hessenberg matrices can be *LU* factored stably in $O(n^2)$ time. For no *LU*-factorizable Hessenberg matrices, the partial pivoting is commonly used. However partial pivoting, e.g. row-interchange operations, can destroy the low rank structure and sparsity of the lower half of the Hessenberg matrices. Thus, some literature has been dedicated to the search of specific (nongaussian) procedures for inverting unreduced Hessenberg matrices [4,5]. Recall that an order $n$ upper Hessenberg matrix $H = (h_{ij})_{i,j=1}^{n}$ is unreduced if the entries on its subdiagonal are nonzero, $h_{i+1,i} \neq 0$, $i = 1, 2, \ldots, n-1$. The particular low rank structure can be exploited in the inversion

* Corresponding author. Tel.: +34 915495700.
*E-mail addresses:* jc.abderraman@upm.es (J. Abderramán Marrero), mu.rachidi@hotmail.fr (M. Rachidi).

http://dx.doi.org/10.1016/j.amc.2015.06.084
0096-3003/

procedure; see e.g. [6,7]. Thus, the matrix inverse of an unreduced upper Hessenberg matrix can be seen as a rank-one perturbation of a strictly upper triangular matrix,

$$H^{-1} = \begin{pmatrix} 0 & t_{12} & t_{13} & \cdots & t_{1n} \\ 0 & 0 & t_{23} & \cdots & t_{2n} \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & t_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} (x_1\ x_2\ x_3\ \cdots\ x_n). \tag{1}$$

The complexity of this decomposition, $O(n^3)$, is equivalent that of back substitution for inverting $H(2:n, 1:n-1)$, the triangular submatrix resulting to delete the first row and the last column of $H$. Note also that the decomposition (1) is not applicable on inverses of reduced Hessenberg matrices.

The Ikebe algorithm [8] computes the entries of the lower half of the inverses of unreduced upper Hessenberg matrices in $O(n^2)$ time. More precisely, it computes the vectors $y$ and $x$ of the inverse decomposition (1). Some important classes of unreduced Hessenberg matrices can be inverted directly using the Ikebe algorithm, e.g. unreduced tridiagonal matrices [8].

**Example 1** (Inverses of upper Hessenberg–Toeplitz matrices)**.** As an illustration, we apply the Ikebe algorithm on nonsingular Hessenberg–Toeplitz matrices, with entries $H_{i,i-1} = h_0 \neq 0$, $H_{ij} = h_{j-i+1}$, for $1 \le i \le j \le n$, and $H_{ij} = 0$, otherwise,

$$H = \begin{pmatrix} h_1 & h_2 & h_3 & \cdots & h_n \\ h_0 & h_1 & h_2 & \ddots & \vdots \\ 0 & h_0 & h_1 & \ddots & h_3 \\ \vdots & \ddots & \ddots & \ddots & h_2 \\ 0 & \cdots & 0 & h_0 & h_1 \end{pmatrix}. \tag{2}$$

For the Hessenberg–Toeplitz matrix (2), the row vector of (1) is $x = x_i = |H_{i-1}|/(-h_0)^{i-1}$, $i = 1, 2, \ldots, n$. Here $|H_{i-1}|$ denotes the determinant of the (left) principal submatrix of order $i-1$. Take $|H_0| = 1$. The row vector $x$ is sufficient for a closed form (1) of the matrix inverse,

$$H^{-1} = \frac{1}{h_0} \begin{pmatrix} 0 & 1 & x_2 & \cdots & x_{n-1} \\ 0 & 0 & 1 & \cdots & x_{n-2} \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \frac{(-h_0)^{n-1}}{|H|} \begin{pmatrix} x_n \\ \vdots \\ x_3 \\ x_2 \\ 1 \end{pmatrix} (1\ x_2\ x_3\ \cdots\ x_n), \tag{3}$$

where the determinant $|H| = (-h_0)^{n-1} x \cdot H(:, n)$, and $H(:, n)$ is the last column of $H$. Hence, it is known that the matrix inverse of Hessenberg–Toeplitz matrices can be computed in $O(n^2)$ time.

For unreduced upper Hessenberg matrices, a lower triangular matrix $L$ with the entries of the lower half of $H^{-1}$ is computed with the Ikebe algorithm. It is interesting for the $LU$ factorization of the matrix inverse. Although, when $H^{-1}$ is not $LU$-factorizable, the matrix $L$ supplied by the Ikebe algorithm is singular. Nevertheless, the superdiagonal of the matrix inverse can be computed without additional computational effort using an expanded Ikebe method. Thus, a general factorization for the matrix inverse, applicable also on reduced Hessenberg matrices, was obtained in [9],

$$H^{-1} = H_L U^{-1}, \tag{4}$$

where the lower Hessenberg matrix $H_L$ is quasiseparable [10], and $U^{-1}$ is an upper triangular matrix with ones on its main diagonal. Such an inverse factorization is not unique and it characterizes the nonsingular upper Hessenberg matrices. Thus in the unreduced case a matrix $H_L$, with the same lower half plus the superdiagonal than $H^{-1}$, is computed with the expanded Ikebe algorithm. A forward substitution scheme for computing $U^{-1}$ was also used in [9].

For nonsingular reduced Hessenberg matrices, the Ikebe algorithm does not work. If $H$ has only a zero on its subdiagonal, $H$ is a $2 \times 2$ block upper triangular matrix. The matrix entries $H_{11}$ and $H_{22}$ are unreduced Hessenberg and these can be inverted easily. We can use the Schur complement for computing the block entry $(H_{12})^{-1}$, i.e. $(H_{12})^{-1} = -(H_{11})^{-1} H_{12} (H_{22})^{-1}$. Note that $H_{21}$ and $(H_{21})^{-1}$ are zero matrices. When the number of zeros on the subdiagonal of $H$ increases, we can design a block procedure for completing the upper half of $H^{-1}$ by using the Schur complement in a reiterated way. As a rule, this block strategy has bad numerical performance.

The rank structure method has been used for inverting some classes of matrices that generalize the Hessenberg matrices; see e.g. [7,11–13].