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Entropies and Heun functions associated with positive linear operators

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ABSTRACT

We consider a parameterized probability distribution $p(x) = (p_0(x), p_1(x), \dots)$ and denote by $S(x)$ the squared l^2-norm of $p(x)$. The properties of $S(x)$ are useful in studying the Rényi entropy, the Tsallis entropy, and the positive linear operator associated with $p(x)$. We show that for a family of distributions (including the binomial and the negative binomial distributions), $S(x)$ is a Heun function reducible to the Gauss hypergeometric function ${}_2F_1$. Several properties of $S(x)$ are derived, including integral representations and upper bounds. Examples and applications are given, concerning classical positive linear operators.

1. Introduction

Let I be a real interval and p_k, $k = 0, 1, \dots,$ non-negative continuous functions defined on I, satisfying $\sum_{k=0}^{\infty} p_k(x) = 1$, $x \in I$. So $p(x) := (p_k(x))_{k \geq 0}$ is a parameterized probability distribution. At the same time, $p(x)$ can be used in order to construct an approximation-theoretic positive linear operator of the form

$$Lf(x) = \sum_{k=0}^{\infty} f(x_k) p_k(x), \quad x \in I,$$

where $x_k \in I$, $k \geq 0$, and f is in a suitable space of functions defined on I.

Particular examples are the binomial distribution (associated with Bernstein operators), the Poisson distribution (associated with Szász–Mirakjan operators), and the negative binomial distribution (corresponding to the Baskakov operators). There is a huge amount of literature on this subject: we refer only to [5].

Let now $S(x) := \sum_{k=0}^{\infty} p_k^2(x)$, $x \in I$.

(A) A first and obvious fact is that $S(x)$ is the squared l^2-norm of the sequence $p(x)$.
(B) In the language of applied probability, $-\log S(x)$ is the Rényi entropy of order 2 [22,23,30] and $1 - S(x)$ is the Tsallis entropy of order 2 [22,27,28].
(C) The "degree of non-multiplicativity" of the operator L can be described in terms of the Chebyshev–Grüss inequalities. Several recent papers have been devoted to this problem; see [3,4,10,11,26] and the references therein. The function $S(x)$ is also involved in this study. More precisely, let L be defined on the space $B(I)$ of all real-valued, bounded functions on I.

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Then (see [11, Theorem 9]),

$$|L(fg)(x) - Lf(x)Lg(x)| \le \frac{1}{2}(1 - S(x))\,\mathrm{osc}(f)\mathrm{osc}(g),$$

where $\mathrm{osc}(f) := \sup\{|f(x_j) - f(x_i)| : i, j \ge 0\}$. Let us remark again that $1 - S(x)$ is the Tsallis entropy of order 2. If for a given x it vanishes, it is easy to see that the functional $A_x : B(I) \to \mathbb{R}$, $A_x(f) = Lf(x)$, is multiplicative and, in fact, A_x is a point-evaluation at some x_k. Thus, to investigate the "degree of non-multiplicativity" of L means, roughly speaking, to see "how far is each A_x from being a point evaluation."

All the three situations (A), (B), (C) mentioned above require a good control on the function $S(x)$.

In this paper we consider a family of distributions $p_n^{[c]}(x) = (p_{n,k}^{[c]}(x))_{k\ge 0}$, introduced in 1957 by Baskakov [6]; see also [7,12,29]. For $c = -1$, $c = 0$, $c = 1$ they correspond respectively to the

- binomial distribution, and the sequence of Bernstein operators,
- Poisson distribution, and the sequence of Szász–Mirakjan operators,
- negative binomial distribution, and the sequence of Baskakov operators.

Let $S_{n,c}(x)$ be the squared l^2-norm of $p_n^{[c]}(x)$. In Section 2 we recall the important results of Berdysheva [7], obtained by using the classical theory of the Gauss hypergeometric function ${}_2F_1$. From these results it is easy to obtain representations of $S_{n,c}$ in terms of ${}_2F_1$ (for $c \ne 0$) and in terms of the modified Bessel function I_0 (for $c = 0$), as well as integral representations of $S_{n,c}$, $c \in \mathbb{R}$.

In Section 3 we extend Berdysheva's approach. By using the classical differential equations satisfied by ${}_2F_1$, respectively I_0, we derive second-order differential equations satisfied by $S_{n,c}$.

An important conclusion is that for $c \ne 0$, $S_{n,c}$ satisfies a Heun equation and so it is a (local) Heun function Hl of a special type: one that is expressible in terms of ${}_2F_1$. Such functions are the object of study in the classical theory of rational reductions of Hl to ${}_2F_1$: see [16]. In Theorem 5 we present a (possibly new) result in this direction.

In Sections 4, 5 and 6 the general results are applied to the special cases $c = -1$, $c = 1$, $c = 0$. Several known results are extended and improved, and new results are given.

We conclude this section by presenting briefly some notation and results concerning the Heun functions; see [8,15–17,24]. For applications of Heun functions in the physical sciences see [13].

The Heun equation is usually written in the form

$$y''(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)y'(x) + \frac{\alpha\beta x - q}{x(x-1)(x-a)}y(x) = 0,$$

where $a \ne 0, 1$ and $\alpha + \beta + 1 = \gamma + \delta + \epsilon$.

If γ is not a nonpositive integer, the solution at $x = 0$ belonging to the exponent zero is analytical.

When normalized by $y(0) = 1$, it is called the *local Heun function*, and is denoted by $Hl(a, q; \alpha, \beta; \gamma, \delta; x)$. It is the sum of a Heun series, which converges in a neighborhood of 0.

We shall need the following transformation formula (see line 3 in Table 2 of [17]):

$$Hl(a, q; \alpha, \beta; \gamma, \delta; x) = \left(1 - \frac{x}{a}\right)^{-\alpha-\beta+\gamma+\delta} Hl(a, q - \gamma(\alpha + \beta - \gamma - \delta); -\alpha + \gamma + \delta, -\beta + \gamma + \delta; \gamma, \delta; x).$$

2. Preliminary results

In this section we recall some results from [7].

Let $c \in \mathbb{R}$. Set $I_c = \left[0, -\frac{1}{c}\right]$ if $c < 0$, and $I_c = [0, +\infty)$ if $c \ge 0$. For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}_0$ the binomial coefficients are defined as usual by

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \quad \text{if } k \in \mathbb{N}, \quad \text{and} \quad \binom{\alpha}{0} := 1.$$

In particular, $\binom{m}{k} = 0$ if $m \in \mathbb{N}$ and $k > m$.

Let $n > 0$ be a real number, $k \in \mathbb{N}_0$ and $x \in I_c$. Define

$$p_{n,k}^{[c]}(x) := (-1)^k \binom{-\frac{n}{c}}{k}(cx)^k(1+cx)^{-\frac{n}{c}-k}, \quad \text{if } c \ne 0,$$

$$p_{n,k}^{[0]}(x) := \lim_{c\to 0} p_{n,k}^{[c]}(x) = \frac{(nx)^k}{k!}e^{-nx}, \quad \text{if } c = 0.$$

Then $\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) = 1$. Throughout the paper we shall suppose that $n > c$ if $c \ge 0$, or $n = -cl$ with some $l \in \mathbb{N}$ if $c < 0$. Under this hypothesis define

$$T_{n,c}(x, y) := \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x)p_{n,k}^{[c]}(y), \quad x, y \in I_c. \tag{1}$$