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Stability and input-to-state stability for stochastic systems and applications

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ABSTRACT

This paper is concerned with establishing some stability and input-to-state stability (ISS) properties in terms of two different measures, h_0 and h, for nonlinear systems of stochastic differential equations of Itô type. To analyze these properties, classical Lyapunov's method and a comparison principle are used. To justify the proposed theoretical results, applications to state estimating systems of Luenberger type and feedforward (or cascade) systems enhanced with numerical examples and simulations are presented.

1. Introduction

To analyze the qualitative behavior of systems, there are many different stability concepts, such as classical Lyapunov stability, partial stability, eventual stability, practical stability, orbital stability, invariance principle of LaSalle, to name a few. These stability notions are adequately unified under a general framework called *stability in terms of two different measures* (or (h_0, h)-based stability). Readers may refer to [12] and many references therein for detailed discussion of these stability concepts. During the last three decades, the (h_0, h)-based stability has been further extended to investigate the qualitative aspects of different systems, such as impulsive differential equations [13,14], deterministic and stochastic switched systems [2], where the jump among the systems modes is controlled by using the average dwell-time switching logic, and, for the stability analysis, multiple Lyapunov functions approach is used. We should also mentioned that in [2,13,14], the authors have established the stability properties by using a comparison principle. In [25], impulsive differential equations with time delay are considered, using Lyapunov method and comparison principle to establish eventual practical stability and quasi-stability, and strong eventual practical stability. For further results on stability in terms of two measures for deterministic systems with and without time delay, one may refer to [3,7,15,24].

System states are often influenced by some environmental disturbances or noises having probabilistic characteristics. If these types of perturbations are considered, the resulting system is called *stochastic systems*. Nowadays, systems with random noises become essential in modeling several natural and human-made phenomena. Due to difficulties arising from the random part of such systems, the interest has been shifted to investigate the stability properties using Lyapunov-like theorems [4,6,16,17].

The notion of input-to-state stability (ISS), presented by Sontag [18,22], deals with the response of asymptotically stable systems to bounded, small input or disturbance regardless of the magnitude of system initial state. During the last two decays and due to its usefulness, ISS has become a central foundation of modern nonlinear feedback and design. It is also considered as a key tool in systems with recursive design and co-prime factorizations, and a connection between the input-output (or external) stability and state (or internal) stability. For further characterizations, implications, and applications of the ISS readers

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may consult [1,10,18–23] and references cited therein. Along this research line, in [8], ISS in terms of different measures for deterministic impulsive and switched systems was also established using Lyapunov method.

As stated earlier, the primary objective of this paper is to establish some new sufficient conditions that guarantee stochastic stability and ISS in terms of different measures for stochastic systems of Itô type using the classical Lyapunov method and comparison principle approach, where in the latter technique it will be shown that some classical stability properties of a scalar, deterministic system imply the corresponding stochastic stability properties (in terms of different measures) of stochastic systems. To demonstrate the effectiveness of the presented analysis, the comparison principle is used to design state estimator systems of Luenberger type. Also, the ISS result is carried over to prove the stability property of feedforward (or cascade) systems consisting of two stochastic subsystems. Moreover, to illustrate the generality of these results some important special cases of these results have also been given throughout the paper. Finally, to enhance these results, some numerical examples and simulations are presented. We should mention that, to the best of the authors' knowledge, these results have not been discussed in the available literature. We also believe that they will have a great contribution to many research fields in sciences or engineering, particularly in the fields of modern nonlinear control design and recursive system design. They also have potential uses in establishing results on stochastic and deterministic systems subject to input disturbance, such as integral-ISS and input-output stability.

The rest of this paper is organized as follows. In Section 2, the problem is formulated and some notations and definitions are given. The main theoretical contributions of the paper are presented in Section 3. The applications and numerical examples are given in Section 4. Finally, the paper results are concluded and summarized in Section 5.

2. Problem formulation

Consider the system of stochastic differential equations of Itô type

$$dx(t) = f(t, x(t))dt + \sigma(t, x(t))dW(t), \tag{1a}$$

$$x(t_0) = x_0, \tag{1b}$$

where, for all $t \in \mathbb{R}_+ = [0, \infty)$, $x(t) \in \mathbb{R}^n$ is the system state and, for some $t_0 \in \mathbb{R}_+$, $x(t_0)$ is the system initial state. For a given matrix $A \in \mathbb{R}^{n\times m}$, the induced norm is defined by $\|A\| = \sqrt{\mathrm{tr}(A^T A)}$ with superscript T being standing for the transpose of the matrix A and tr being standing for the trace of a square matrix. Denote by $\mathcal{C}([a, b], \mathbb{D})$ the space of continuous functions mapping $[a, b]$, with $a < b$ for any $a, b \in \mathbb{R}_+$, into $\mathbb{D}$, for some open set $\mathbb{D} \subseteq \mathbb{R}^n$. Also, denote by $\mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ the space of all real-valued functions $V(t, x)$ defined on $\mathbb{R}_+ \times \mathbb{R}^n$ such that they are continuously differentiable once in t and twice in x. For instance, if $V(t, x) \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$, then we have

$$V_t(t, x) = \frac{\partial V(t, x)}{\partial t}, \quad V_x(t, x) = \left(\frac{\partial V(t, x)}{\partial x_1}, \dots, \frac{\partial V(t, x)}{\partial x_n}\right), \quad V_{xx}(t, x) = \left(\frac{\partial^2 V(t, x)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Let $W(t) \in \mathbb{R}^m$ be a Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ on Ω that satisfies the usual condition (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$), and $\mathbb{P}$ be a probability measure on the probability space $(\Omega, \mathcal{F})$. Moreover, for a given filtration $\{\mathcal{F}_t : a \le t \le b\}$, W is assumed to be $\mathcal{F}_t$-measurable (or $\mathcal{F}_t$-adapted) and, for any $s \le t$, the increment $W(t) - W(s)$ is independent of the σ-algebra $\mathcal{F}_s$. Denote by $\mathbb{E}[u]$ the mathematical expectation (or mean) of a random process u. For $p \ge 1$, let $\mathcal{L}_{ad}(\Omega, L^p[a, b])$ denote the class of random processes that are $\mathcal{F}_t$-adapted and almost all their sample paths are Riemann integrable. In (1), the drift vector $f(t, x)$ mapping $\mathbb{R}_+ \times \mathbb{D}$ into $\mathbb{R}^n$ is a class $\mathcal{L}_{ad}(\Omega, L^1[a, b])$ function and the diffusion matrix $\sigma(t, x)$ mapping $\mathbb{R}_+ \times \mathbb{D}$ into $\mathbb{R}^{n\times m}$ is a class $\mathcal{L}_{ad}(\Omega, L^2[a, b])$ function. To ensure that system (1) possesses a unique solution $x = x(t, t_0, x_0)$, we assume that the $f(t, x)$ and $\sigma(t, x)$ are piecewise continuous in $t \in [a, b]$ and satisfy Lipschitz condition in $x \in \mathbb{D}$. For the system to admit a trivial solution $x(t) \equiv 0$, we assume that $f(t, 0) \equiv 0$ and $\sigma(t, 0) \equiv 0$, for all $t \ge t_0$.

The following classes of monotonic functions will be used throughout this paper to analyze the stability properties, where readers may refer to [5,9] for more detailed discussion.

$\mathcal{K} = \{a \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}_+) : a(0) = 0$ and it is strictly increasing$\}$;
$\mathcal{K}_1 = \{a \in \mathcal{K}$ and it is convex$\}$;
$\mathcal{K}_2 = \{a \in \mathcal{K}$ and it is concave$\}$;
$\mathcal{L} = \{a \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}_+) : a$ is strictly decreasing and $\lim_{s\to\infty} a(s) = 0\}$;
$\mathcal{KL} = \{\beta \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}_+; \mathbb{R}_+) : \beta(\cdot, s) \in \mathcal{K}\ \forall$ fixed s and $\beta(t, \cdot) \in \mathcal{L}$ for each fixed $t\}$;
$\mathcal{CK} = \{\varphi \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}_+; \mathbb{R}_+) : \varphi(t, \cdot) \in \mathcal{L}$ for each fixed $t\}$.

To analyze the (h_0, h)-based stability, a certain class of $\mathcal{F}_t$-adapted random processes that are used as measure functions is defined by

$$\Gamma = \{h \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+) : \inf_{(t,x(t))} h(t, x(t)) = 0\}$$

and the open tube of radius $\lambda > 0$ is defined by

$$S(h, \lambda) = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid \mathbb{E}[h(t, x(t))] < \lambda\},$$