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# Two (2+1)-dimensional expanding dynamical systems associated to the mKP hierarchy

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ABSTRACT

An isospectral problem with a parameter $\epsilon$ is presented, for which a modified KdV (mKdV) hierarchy is easily derived from the Tu scheme. Then two different expanding integrable models are obtained, respectively. The main purpose for this focuses on deriving a (2+1)-dimensional mKdV hierarchy (called the mKP hierarchy) and the corresponding different (2+1)-dimensional expanding models (including the linear and nonlinear), whose Hamiltonian structures are obtained by employing an identity developed by us in the paper. As the reduced consequences, the linear and nonlinear (2+1)-dimensional expanding models of the mKP equation are generated.



## 1. Introduction

Given an integrable system

$$u_t = K(u), \tag{1}$$

we try to find another integrable system

$$v_t = S(u, v) \tag{2}$$

so that the coupled system

$$\begin{cases} u_t = K(u), \\ v_t = S(u, v) \end{cases} \tag{3}$$

is still integrable. The idea was early proposed by Fuchssteiner [1] when he investigated the Virasoro algebras. Eq. (3) was called an integrable coupling of the system (1). Later, Ma [2] applied the perturbation method to investigate the integrable couplings of the KdV equation. We can regard the integrable coupling (3) as the case where we try to find the integrable system (2) under the constraint (1). Thus, Eq. (2) is a variable-coefficient equation whose coefficients $u$ and $u_x, u_{xx}, \ldots$ are presented in Eq. (1). Therefore, the course for seeking for the integrable coupling (3) can be regarded as researching for variable-coefficient differential equations. In addition, by reduction of Eq. (2), we can obtain new integrable equations. Since it is a complicated work to employ the perturbation method for generating Eq. (3), we would like to employ Tu's work [3] to investigate integrable systems with the help of Lie algebras along with square matrices. The above procedure for generating integrable systems was called the Tu scheme

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[4]. With the aid of the Tu scheme, some interesting integrable equations and the related properties of nonlinear equations were obtained, such as the results in Refs. [5–12]. Guo and Zhang [13] once constructed an integrable model to unite the well-known AKNS hierarchy and the KN hierarchy; the fact indicates the significance for the study of the integrable couplings. In order to obtain integrable couplings (3), it is most important to find the suitable Lie algebras and the corresponding loop algebras. We usually require the Lie algebra $G$ that satisfies the decomposition as follows:

$$G = G_1 \oplus G_2, [G_1, G_2] \subset G_2, \tag{4}$$

where $G_1$ and $G_2$ are two subalgebras of $G$. The symbol $[\,\cdot, \cdot]$ stands for a commutator of the Lie algebra $G$. Furthermore, we still require the Lie algebra $G$ be semi-simple. Under the condition (4), we constructed some Lie algebras and obtained a few of expanding integrable models, that is, integrable couplings. In Refs. [14–16], we introduced a few kinds of Lie algebras satisfying (4) and obtained some integrable couplings and their some properties.

Concerning the generating (2+1)-dimensional integrable hierarchies, one has proposed some methods. For example, Ablowitz et al. [17] applied the self-dual Yang–Mills equations to obtain their equivalent representations, from which some well-known (2+1)-dimensional integrable systems, such as the KP equation and the N-wave equation, were obtained. By following the idea, Zhang and Tam [18] obtained a generalized variable-coefficient Burgers equation and a (2+1)-dimensional integrable coupling of a new (2+1)-dimensional integrable system. Another example for generating (2+1)-dimensional integrable hierarchies of evolution equations was proposed by Tu et al. [19], which is called the TAH scheme. The scheme needs us to introduce a residue operator and the operator $\xi$ to generate the (2+1)-dimensional hierarchies by using the Tu scheme. Although the TAH scheme was proposed early in 1990, few people further go on investigating and improving it. Therefore, we want to discuss the scheme to investigate some new higher-dimensional systems and their Hamiltonian structures in the paper. Now we recall some associated notations.

Let $\mathcal{A}$ be an associative algebra over the field $\mathcal{R}$. An operator, $\partial : \mathcal{A} \to \mathcal{A}$, is introduced which satisfies that

$$\partial(\alpha f + \beta g) = \alpha(\partial f) + \beta(\partial g), \partial(fg) = (\partial f)g + f(\partial g),$$

where $\alpha, \beta \in \mathcal{R}$; $f, g \in \mathcal{A}$. We again introduce an associative algebra $\mathcal{A}[\xi]$ which consists of the pseudo-differential operator $\sum_{-\infty}^{N} a_i \xi^i$, where the coefficient $a_i \in \mathcal{A}$, and $\xi$ stands for an operator given by

$$\xi f = f\xi + (\partial_y f), f \in \mathcal{A}, \tag{5}$$

from which one can verify that [19]

$$\xi^n f = \sum_{i \geq 0} \binom{n}{i} (\partial^i f)\xi^{n-i}, n \in \mathbf{Z}, \tag{6}$$

In the associative algebra $\mathcal{A}[\xi]$, a residue operator is defined as

$$R : \mathcal{A}[\xi] \to \mathcal{A}, \quad R\Big(\sum a_i \xi^i\Big) = a_{-1}. \tag{7}$$

It is remarkable that the linear extension of the operator $\xi$ introduced in [19] is equivalent to two central independent (matrix) extensions over independent variables of matrix loop algebras, which was presented in Ref. [20]. In addition, if the associative algebra $\mathcal{A}$ is $n$ dimensional, the corresponding operator algebra $\mathcal{A}[\xi]$ is also $n$ dimensional, which is denoted by $\mathcal{A}_n[\xi]$.

In this paper, we shall deduce two different (1+1)-dimensional integrable couplings of the mKdV hierarchy. Based on this and the TAH scheme, we further construct two new (2+1)-dimensional integrable couplings of the mKP hierarchy, whose Hamiltonian structures are generated by an improved variational identity.

## 2. The mKdV hierarchy and two expanding integrable models

### 2.1. The mKdV hierarchy

Set $A_1 = \text{span}\{h, e, f\}$, where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

A loop algebra of the Lie algebra $A_1$ is denoted by

$$\tilde{A}_1 = \text{span}\{h(n), e(n), f(n)\},$$

where $h(n) = h\lambda^n, e(n) = e\lambda^n, f(n) = f\lambda^n$.

Take

$$U = h(1) + ue(0) - \epsilon u f(0) \in \tilde{A}_1,$$