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# Extended two dimensional equation for the description of nonlinear waves in gas–liquid mixture

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ABSTRACT

We consider a system of equations for the description of nonlinear waves in a liquid with gas bubbles. Taking into account high order terms with respect to a small parameter, we derive a new nonlinear partial differential equation for the description of density perturbations of mixture in the two-dimensional case. We investigate integrability of this equation using the Painlevé approach. We show that traveling wave reduction of the equation is integrable under some conditions on parameters. Some exact solutions of the equation derived are constructed. We also perform numerical investigation of the nonlinear waves described by the derived equation.



## 1. Introduction

A liquid with gas bubbles is a complex dissipative and dispersive nonlinear media. Nonlinear character of waves in such medium brings essential difficulties for investigation, although there are some interesting properties of wave processes in gas–liquid mixture and mathematical models for the description of such systems occur widely in different sciences: chemistry, biology, physic, etc. (see [1–3]).

For the first time nonlinear evolution equations like Burgers, Korteweg–de Vries and Burgers–Korteweg–de Vries were obtained for the description of long weakly nonlinear waves in a gas–liquid mixture in works [4–6] for the one-dimensional case. The three-dimensional case was considered in work [7], but only first-order terms in an asymptotic series have been taken into account. On the other hand, considering high-order corrections in asymptotic expansions, we are able to obtain more complicated nonlinear equations. It allows us to describe wave processes more accurate than in [7]. Besides, we can discover some new physical effects. In work [8] models for non-linear waves in a gas–liquid mixture were generalized, taking into account higher order terms with respect to small parameters. Models of work [8] take into consideration an interphase heat transfer, surface tension and weak liquid compressibility, although only one-dimensional case is considered. Thus, it is interesting to study long weakly nonlinear waves in a liquid with a gas bubbles in two-dimensional case, taking into consideration both high order terms in the asymptotic expansions and physical properties mentioned above.

Here we derive a new nonlinear partial differential equation for the description of long weakly nonlinear two-dimensional waves in a bubbly gas–liquid mixture. We consider waves propagating in a certain direction. We assume that perturbations in perpendicular directions are less essential but we take them into account. We also take into account high order terms in the

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asymptotic expansions, interphase heat transfer, surface tension and weak liquid compressibility. We also investigate equation derived analytically and numerically. To the best of our knowledge, this equation has not been obtained and investigated before.

In order to investigate integrability of the nonlinear equation we apply the Painlevé approach. It is shown that the equation does not have the Painlevé property in general case. However solitary wave solutions are constructed by means of the truncated expansion method. Using traveling wave variables it is shown that the equation passes the Painlevé test under some conditions on parameters. With the Hopf–Cole transformations the equation is linearized and its solutions are obtained in different forms. Nonlinear waves described by the equation are also investigated numerically using the spectral method. It is shown that this method has good accuracy and stability.

The rest of this work is organized as follows. In Section 2 we derive a nonlinear partial differential equation for the description of waves in gas–liquid mixture, taking into consideration second order terms with respect to the small parameters. In Section 3 we apply the Painlevé approach to investigate integrability of the equation. In Section 4 the new nonlinear equation is investigated using traveling wave variables. It is shown that the equation is integrable under some conditions on parameters. In Section 5 we present the results of the numerical simulation of waves, described by the equation. In Section 6 we briefly discuss our results.

## 2. Extended equation for the description of waves in a liquid with gas bubbles in two-dimensional case

In this section we obtain a two-dimensional nonlinear equation for the description of waves in a liquid with gas bubbles. We use the system of equations for the description of waves in bubbly liquid, presented in [7]. We suppose that the gas–liquid mixture is a homogeneous medium with an average pressure and temperature. We assume that the liquid is incompressible and gas bubbles are spherical. We do not consider destruction, formation, interaction and coalescence of bubbles. We suppose that the total amount of gas in a bubble and the amount of gas bubbles in unit of mass of liquid are constant. Gas in bubble is an ideal and the pressure in bubble is described by the politropic law. Liquid viscosity is considered only on the interphase boundary. Taking into account assumptions mentioned above, the following system of equations for the description of waves in liquid with gas bubbles is used (see [7] )

$$\begin{aligned}
&\frac{\partial\tilde\rho}{\partial\tau}+\nabla\tilde{\mathbf{u}}+\nabla(\tilde\rho\tilde{\mathbf{u}})=0,\\
&(1+\tilde\rho)\left(\frac{\partial\tilde{\mathbf{u}}}{\partial\tau}+\tilde{\mathbf{u}}\nabla\tilde{\mathbf{u}}\right)+\frac{1}{\alpha}\nabla\tilde p=0,\\
&\tilde p=\alpha\tilde\rho+\alpha_1\tilde\rho^2+\alpha_2\tilde\rho^3+\beta\tilde\rho_{\tau\tau}-(\beta_1+\beta_2)\tilde\rho\tilde\rho_{\tau\tau}-\left(\beta_1+\frac{3}{2}\beta_2\right)\tilde\rho_\tau^2+\varkappa\tilde\rho_\tau+\varkappa_1\tilde\rho\tilde\rho_\tau.
\end{aligned}\tag{1}$$

Here $\tilde p,\ \tilde\rho,\ \tilde{\mathbf{u}}$ are the non-dimensional pressure, density and velocity of the mixture correspondingly, $\nabla$ is the two-dimensional Nabla operator, $\xi,\eta$ are Cartesian coordinates and $\tau$ is the time; $\alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2,\varkappa,\varkappa_1$ are non-dimensional parameters [7].

For the derivation of an equation for the description of nonlinear waves, we use the reductive perturbation method (see e.g. [9–13]). Let us introduce 'slow' variables

$$x=\epsilon(\xi-\tau),\ y=\epsilon^{\frac{3}{2}}\delta\eta,\ t=\epsilon^2\tau.\tag{2}$$

We suppose that perturbations in $x$ direction are more essential then in $y$. We chose power of $\epsilon$ in 'slow' variables in order to obtain equations for the case of dissipation main influence. We search for the solution of system (1) in the form of asymptotic series:

$$\begin{aligned}
&\tilde u^{(1)}=\epsilon u_1^{(1)}+\epsilon^2u_2^{(1)}+\cdots, &&\tilde u^{(2)}=\epsilon u_1^{(2)}+\epsilon^2u_2^{(2)}+\cdots,\\
&\tilde\rho=\epsilon\rho_1+\epsilon^2\rho_2+\cdots, &&\tilde p=\epsilon p_1+\epsilon^2p_2+\cdots.
\end{aligned}\tag{3}$$

Substituting (2) and (3) into (1) and collecting coefficients at $\epsilon^0$ we obtain

$$u_1^{(1)}=\rho_1,\qquad p_1=\alpha\rho_1.\tag{4}$$

Collecting coefficients at the same powers of $\epsilon$ and using (4) we have the following equations:

$$\begin{aligned}
&\rho_{1t}-\rho_{2x}+u_{2x}^{(1)}+(\rho_1u_1^{(1)})_x+\epsilon\delta u_{1y}^{(2)}=0,\\
&u_{1t}^{(1)}-u_{2x}^{(1)}+u_1^{(1)}u_{1x}^{(1)}+\rho_{2x}+\frac{2\alpha_1}{\alpha}\rho_1\rho_{1x}-\frac{\varkappa}{\alpha}\rho_{1xx}-\rho_1u_{1x}^{(1)}\\
&\quad+\epsilon\left(\rho_1u_{1t}^{(1)}+\rho_1u_1^{(1)}u_{1x}^{(1)}+\frac{\alpha_2}{\alpha}(\rho_1^3)_x+\frac{\beta}{\alpha}\rho_{1xxx}+\frac{\varkappa}{\alpha}\rho_{1tx}-\frac{\varkappa_1}{\alpha}(\rho_1\rho_{1x})_x\right)=0,
\end{aligned}\tag{5}$$

$$u_{1x}^{(2)}=\delta\rho_{1y}+\epsilon\left(u_{1t}^{(2)}+u_1^{(2)}u_{1x}^{(1)}\frac{\delta\alpha_1}{\alpha}\rho_{1y}^2-\frac{\delta\varkappa}{\alpha}\rho_{1xy}-\rho_1u_{1x}^{(2)}\right).\tag{6}$$

Differentiating (5) and (6) with respect to $x$ and $y$ correspondingly and using obtained relations to avoid velocity, we get: