Applied Mathematics and Computation 268 (2015) 596–608

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A branch-and-cut algorithm for a class of sum-of-ratios problems

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ARTICLE INFO

Keywords:
Global optimization
Fractional programming
Semi-infinite optimization
Cutting plane
Branch-and-bound
Branch-and-cut

ABSTRACT

The problem of maximizing a sum of concave–convex ratios over a convex set is addressed. The projection of the problem onto the image space of the functions that describe the ratios leads to the equivalent problem of maximizing a sum of elementary ratios subject to a linear semi-infinite inequality constraint. A global optimization algorithm that integrates a branch-and-bound procedure for dealing with nonconcavities in the image space and an efficient relaxation procedure for handling the semi-infinite constraint is proposed and illustrated through numerical examples. Comparative (computational) analyses between the proposed algorithm and two alternative algorithms for solving sum-of-ratios problems are also presented.

1. Introduction

Initially restricted to problems with only one ratio, the field of *fractional programming* [1] has broadened its scope in the last decades by incorporating the analysis of multi-ratio problems. Progresses in multi-ratio fractional optimization have led to efficient global optimization algorithms for the problem of maximizing the smallest of several ratios, and to effective algorithms for maximizing or minimizing a sum-of-ratios [2]. The former problem is considerably more tractable than the later, as many of the properties of concave–convex single-ratio problems, such as the semistrictly quasiconcavity of its objective function [3], are inherited by the problem of maximizing the smallest concave–convex ratio.

Differently from the smallest of several concave–convex ratios, the sum of concave–convex ratios is generally not quasiconcave, which implies that it may have a local maximizer which is not a global maximizer. In fact, it has been shown that even the problem of maximizing the sum of a concave function and a concave–convex ratio is *NP*–complete [4]. However, in spite of the current limitations of its theoretical basis, sum-of-ratios problems, specially the linear sum-of-ratios problem, understood as the problem of maximizing a sum of ratios of linear functions over a polytope, have attracted considerable interest [2].

Most global optimization approaches that address sum-of-ratios problems explore the concept of *image* (or *outcome*) *space*. The image space is defined by a mapping which associates an additional variable with each ratio [5,6], or additional variables with the numerator and denominator of each ratio [7–10].

There are theoretical and practical advantages in considering sum-of-ratios problems in the image space. Firstly, image-space analyses lead to correspondences between the optimal solutions of sum-of-ratios problems and *Pareto-optimal* or *efficient*

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http://dx.doi.org/10.1016/j.amc.2015.06.089
0096-3003/

solutions of certain related multiobjective optimization problems [6,11]. Secondly, sum-of-ratios problems become more tractable in the image space. As the nonconcavity of the problem (preserved in the image space) is expressed in terms of the additional image-space variables only, global optimization methods, and particularly branch-and-bound methods [12], find ideal implementation conditions [8,10,13–17]. Additionally, the usual assumptions about the ratios induce an implicit convex feasible set in the image space, which can be iteratively described by outer approximation algorithms [13].

Outer approximation and branch-and-bound techniques are also present in the approach proposed in this paper for the problem of maximizing a sum of concave–convex ratios over a convex set. However, in contrast with most existing image-space approaches, we introduce a decomposition between the (eventually many) original decision variables and the (usually not many) additional image-space variables of the problem.

By projecting the problem onto the image space, we obtain the equivalent problem of maximizing a sum of elementary ratios (of image-space variables) subject to a linear semi-infinite inequality constraint. Then we demonstrate that the equivalent problem is globally solvable by an optimization algorithm that integrates two other algorithms: a rectangular branch-and-bound algorithm, responsible for dealing with nonconcavities in the image space, and a cutting plane algorithm, responsible for handling the semi-infinite inequality constraint. The bounding procedure of the first algorithm is based on the overestimation of each elementary ratio by its concave envelope over a given rectangle [8,18].

Computational tests presented in this paper show that a relatively small sample of the semi-infinite constraint determined by an equally small number of bisections of the feasible region suffices for solving sum-of-ratios problems through the branch-and-cut approach proposed. Computational tests also show that the efficiency of the algorithm increases (when compared to a standard branch-and-bound algorithm) as the number of ratios, variables and constraints increase, and that the algorithm compares favorably with an alternative and influential image-space algorithm for minimizing sums of linear ratios over polytopes.

The paper is organized as follows. In Section 2 we develop a decomposition approach for the problem of maximizing a sum of concave–convex ratios over a convex set. The equivalent problem of maximizing a sum of elementary ratios subject to a linear semi-infinite inequality constraint is established, as well as the structure and properties of the cutting plane algorithm used for handling the semi-infinite constraint. We then outline a rectangular branch-and-bound algorithm for solving the problem of maximizing a sum of elementary ratios subject to a finite number of linear inequality constraints, which corresponds to a typical iteration of the branch-and-cut algorithm. Computational tests with the resulting global optimization algorithm are reported in Section 3. Conclusions are presented in Section 4.

2. Theory

The notations used throughout this paper are fairly standard. The set of all n-dimensional real vectors is represented as $\mathbb{R}^n$; the set of all non-negative real vectors as $\mathbb{R}^n_+$. Inequalities are meant to be componentwise: given $x, y \in \mathbb{R}^n$, then $x \geq y$ implies that $x_i \geq y_i, i = 1, 2, \ldots, n$. The Euclidean norm of $x \in \mathbb{R}^n$ is denoted as $\|x\|$. The symbol $:=$ means *equal by definition to*. Finally, $v^\star(P)$ represents the (global) optimal value of any given optimization problem (P).

Let $\mathcal{X} \subset \mathbb{R}^n$ be a nonempty compact convex set, $f_i \colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, p$, positive, convex functions on $\mathcal{X}$ and $g_i \colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, p$, positive, concave functions on $\mathcal{X}$. The global optimization approach proposed in this paper applies to sum-of-ratios problems both in the minimization (convex–concave) sense,

$$(P_N) \left| \begin{array}{ll} \underset{x}{\text{minimize}} & \displaystyle\sum_{i=1}^{p} \frac{f_i(x)}{g_i(x)} \\ \text{subject to} & x \in \Omega, \end{array} \right.$$

and in the maximization (concave–convex) sense,

$$(P_X) \left| \begin{array}{ll} \underset{x}{\text{maximize}} & \displaystyle\sum_{i=1}^{p} \frac{g_i(x)}{f_i(x)} \\ \text{subject to} & x \in \Omega. \end{array} \right.$$

Despite all the previous convexity, concavity and positivity assumptions, the sum-of-ratios problems (P_N) and (P_X) are both nonconvex, that is, they may have a local optimal solution that is not globally optimal. In Theorem 1 we show that their optimal values satisfy a simple inequality.

Theorem 1. *The optimal values of* (P_N) *and* (P_X) *satisfy the inequality*

$$v^\star(P_X)v^\star(P_N) \geq p^2. \tag{1}$$

Proof. Using the positivity of $f_i(x), g_i(x), i = 1, 2, \ldots, p$, over Ω, and invoking the mean-harmonic inequality, we obtain

$$\frac{1}{p}\sum_{i=1}^{p} \frac{g_i(x)}{f_i(x)} \geq \frac{p}{\displaystyle\sum_{i=1}^{p} \frac{f_i(x)}{g_i(x)}} \tag{2}$$

for all $x \in \Omega$. The inequality (1) is then established by taking the maximum over Ω on both sides of (2). $\square$