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Some techniques for solving absolute value equations

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A B S T R A C T

In this paper, we introduce and analyze two new methods for solving the NP-hard absolute value equations (AVE) $Ax - |x| = b$, where A is an arbitrary $n \times n$ real matrix and $b \in R^n$, in the case, singular value of A exceeds 1. The comparison with other known methods is carried to show the effectiveness of the proposed methods for a variety of randomly generated problems. The ideas and techniques of this paper may stimulate further research.

1. Introduction

In this section, we introduce the absolute value equations (AVE) as

$$Ax - |x| = b, \tag{1}$$

where $A \in R^{nxn}$, $b \in R^n$, and $|x|$ denotes the component-wise absolute value of vector $x \in R^n$.

A slightly more general form of the AVE was introduced in [34] and investigated in a more general context in [28]. It was proved in [25] that the AVE (1) can be equivalently reformulated as a linear complementarity problem (LCP), which is one of the most important problems in the applied sciences and engineering, see [13–15,29–31].

The equivalence between the absolute equations and the complementarity problems can be exploited to suggest several iterative methods for solving the absolute value equations. It is also well known fact that the complementarity problems are also equivalent to the variational inequalities. This means that one can use the technique of the variational inequalities to suggest several iterative methods for solving the absolute value equations. In spite of the extensive activities going on in this field, it is high time to use the variational inequalities in solving the absolute value equations. For the formulation, numerical methods, sensitivity analysis, dynamic systems and other aspects of the variational inequalities, see, for example, Noor [3–6,10] and the references therein.

Recently a semismooth Newton method [26] is proposed for solving the absolute value equations. It shows that the semismooth Newton iterates are well defined and bounded when the singular values of A exceed 1. In [41], a generalized Newton

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method, which has global and finite convergence, was proposed for solving the AVE. The method utilizes both the semismooth and the smoothing Newton steps, in which the semismooth Newton step guarantees the finite convergence and the smoothing Newton step contributes to the global convergence.

Yong considered a particle swarm optimization (PSO) to AVE based on aggregate function [39], and the Harmony Search (HS) algorithm for solving AVE in [40]. Noor et al. [9,12] have suggested and analyzed an iterative method for solving the absolute value equations using the minimization technique, in the case where the coefficient matrix is symmetric. Also Rohn [33] described an algorithm which in a finite (but exponential) number of steps computes all solutions of an absolute value equations. Also, several methods were proposed to obtain solution of (1) [27,28,40].

In this paper, we introduce two new methods for solving the absolute value equations. We compare these new methods with known methods for accuracy and calculation time on various randomly generated problems.

This paper is organized as follows. The theoretical results for absolute value equations are described in Section 2. In Section 3, we describe new techniques for solving absolute value equations. Some important known techniques for solving AVE are reviewed in Section 4. In Section 5, we compare all methods for accuracy and calculation time on various randomly generated problems. Readers are encouraged to find the applications of the absolute value equations in pure and applied sciences, which is another direction for future research.

We now describe our notation. For $x \in R^n$, $\|x\|$ denotes 2−norm and we denote $|x|$ the vector in R^n of absolute values of components of x. For $x \in R^n$, sgn(x) denotes a vector with components equal to $1, 0, -1$ depending on whether the corresponding component of x is positive, zero or negative. The diagonal matrix $D(x)$ is defined as

$$D(x) = \partial|x| = \mathrm{diag}(\mathrm{sgn}(x)), \tag{2}$$

(the diagonal matrix corresponding to sgn(x)), where $\partial|x|$ represents the generalized Jacobian of $|x|$ based on a subgradient, see [32,33].

2. Theoretical results

In this section, we recall the some basic results showing the existence of a solution of the absolute value equations.

Proposition 2.1.

1. *The AVE (1) is uniquely solvable for any $b \in R^n$, if the singular values of A exceed 1.*
2. *If 1 is not an eigenvalue of A and the singular values of A are greater or equal to 1, then the AVE (1) is solvable if $\{(A+I)x - b \geq 0, (A-I)x - b \geq 0\} \neq \emptyset$.*
3. *The AVE (1) is uniquely solvable for any $b \in R^n$ if $\|A^{-1}\| < 1$.*

Proof. See [25]. □

Proposition 2.2. *Let $0 \neq b \geq 0$ and $\|A\| < 1$. Then the AVE (1) has no solution.*

Proof. See [25]. □

Proposition 2.3. *The AVE (1) has no solution for any A, b such that: $r \geq A^T r \geq -r$, $b^T r > 0$, $r \in R^n$.*

Proof. See [25]. □

Now, we show that the AVE (1) is in fact equivalent to a bilinear program (an optimization problem with an objective function that is the product of two affine functions). To do this, we quote the following lemma from [25].

Lemma 2.4. *The AVE (1) is equivalent to the bilinear program as follows:*

$$\begin{aligned} 0 = \min_{x\in R^n} & ((A+I)x - b)^T((A-I)x - b), \\ & (A+I)x - b \geq 0, \\ & (A-I)x - b \geq 0. \end{aligned} \tag{3}$$

We note that if $\lambda_{\min}(A^TA) \geq 1$, then the Hessian of the objective function of (3) is $2(A^TA - I)$, which is a positive semidefinite matrix (*mineig* denotes the least eigenvalue). Therefore we conclude that the bilinear quadratic program 3 is a convex problem and we have the following theorem.

Theorem 2.5. *If $\lambda_{\min}(A^TA) \geq 1$ then the bilinear quadratic program 3 is a convex problem and is equivalent to the following convex system:*

$$\begin{aligned} ((A+I)x - b)^T((A-I)x - b) \leq 0, \\ (A+I)x - b \geq 0, \\ (A-I)x - b \geq 0. \end{aligned} \tag{4}$$