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Approximative solutions to difference equations of neutral type

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ABSTRACT

Asymptotic properties of solutions to difference equations of the form

$$\Delta^m(x_n - u_n x_{n-k}) = a_n f(x_n) + b_n$$

are studied. Replacing the sequence u by its limit and the right side of the equation by zero we obtain an equation which we call the fundamental equation. First we investigate the space of all solutions of the fundamental equation. We show that any such solution is a sum of a polynomial sequence and a product of a geometric sequence and a periodic sequence. Next, using a new version of the Krasnoselski fixed point theorem and the iterated remainder operator, we establish sufficient conditions under which a given solution of the fundamental equation is an approximative solution to the above equation. Our approach, based on the iterated remainder operator, allows us to control the degree of approximation. In this paper we use $o(n^s)$, for a given nonpositive real s, as a measure of approximation.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of positive integers, the set of all integers and the set of real numbers, respectively. For $p, k \in \mathbb{Z}$ let

$$\mathbb{N}(p) = \{p, p+1, \ldots\}, \qquad \mathbb{N}(p,k) = \{p, p+1, \ldots, k\}.$$

Let $m \in \mathbb{N}$. In this paper we consider the difference equation of the form

$$\Delta^m(x_n - u_n x_{n-k}) = a_n f(x_n) + b_n \tag{E}$$

$$a_n, b_n, u_n \in \mathbb{R}, \quad k \in \mathbb{Z}, \quad f\colon \mathbb{R} \to \mathbb{R}.$$

By a solution of (E) we mean a sequence $x\colon \mathbb{N}(0) \to \mathbb{R}$ satisfying (E) for all large n.

Asymptotic properties of solutions to neutral equations were investigated in many papers, see, for example, [1–9], [11–13]. In this paper we investigate the existence of approximative solutions. Let $y\colon \mathbb{N}(0) \to \mathbb{R}$ and $s \in (-\infty, 0]$. We say that y is $o(n^s)$-approximative solution of (E) if there exists a solution x of (E) such that

$$y = x + o(n^s). \tag{1}$$

Writing (1) in the form

$$x = y + o(n^s)$$

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one can say that x is a solution with prescribed asymptotic behavior.

The existence of solutions with prescribed asymptotic behavior to difference equations of neutral type is the theme of many papers. For example, the existence of non-oscillatory or positive or bounded solutions is investigated in the papers [3], [5–9] or in [21]. Our approach is based on an idea analogous to the idea used in the theory of ordinary difference equations, for example, of the form

$$\Delta^m x_n = a_n f(x_n) + b_n. \tag{2}$$

If sequences a, b are 'sufficiently small' and f is 'sufficiently regular', then some solutions of the equation $\Delta^m y_n = 0$ are approximative solutions to (2) (see [15], [16] or [18]). In this paper, in Section 3, we solve the equation

$$\Delta^m (y_n - \lambda y_{n-k}) = 0 \tag{3}$$

which we call the fundamental equation of neutral type. We show, assuming $|\lambda| \neq 1$, that any solution to (3) is of the form

$$y_n = \varphi(n) + \omega_n \mu^n$$

where φ is a polynomial sequence of degree $\le m$, ω is a $2|k|$-periodic sequence and

$$\mu = \sqrt[k]{|\lambda|}.$$

Next, in Theorem 4.1, assuming $u_n \to \lambda$, we establish conditions under which a given solution y of (3) is an approximative solution to (E). In particular, taking $\omega = 0$, we obtain asymptotically polynomial solutions or, if $\omega > 0$ and $\mu > 1$ we obtain positive approximative solution with exponential growth.

In the study of existence of solutions with prescribed asymptotic behavior to difference equations of neutral type the Krasnoselski fixed point theorem is often used. This theorem is applicable to compact and convex subsets of Banach spaces. Unfortunately, the 'sup' norm is not a norm on the space $\mathbb{R}^{\mathbb{N}}$ of all sequences (it takes the value ∞). We introduce the notion of a regional norm which can take the value ∞ and the notion of a Banach regional space. Then $\mathbb{R}^{\mathbb{N}}$ with 'sup' norm is a Banach regional space. Next we obtain the 'regional' version of the Schauder fixed point theorem and the Krasnoselski fixed point theorem. These theorems are applicable to any compact and convex subsets of Banach regional spaces. In the proof of Theorem 4.1, which is our main result, we use our version of the Krasnoselski fixed point theorem.

In the cycle of papers [10], [14–20] a new method in the study of asymptotic properties of solutions to difference equations is presented. This method, based on using the iterated remainder operator, allows us to control the degree of approximation. In this paper we use $o(n^s)$, for a given nonpositive real s, as a measure of approximation.

The paper is organized as follows. In Section 2, we introduce notation and terminology. Next, in Section 2.1, we present the iterated remainder operator and its basic properties - these results are taken from [18]. Next, in Section 2.2, we obtain our version of the Schauder fixed point theorem and the Krasnoselski fixed point theorem. In Section 3, we compute the space of all solutions to the fundamental equation of neutral type. In Section 4, in Theorem 4.1, we obtain our main result.

2. Notation and terminology

Let $\mathbb{N}_0 = \mathbb{N}(0)$ denote the set of nonnegative integers. The space of all sequences $x: \mathbb{N}_0 \to \mathbb{R}$ we denote by SQ.

The space of all bounded sequences $x \in$ SQ we denote by BS.

If x, y in SQ, then xy denotes the sequence defined by pointwise multiplication

$$xy(n) = x_n y_n.$$

Moreover, $|x|$ denotes the sequence defined by

$$|x|(n) = |x_n|.$$

We use the symbols 'big O' and 'small o' in the usual sense but for $a \in$ SQ we also regard $o(a)$ and $O(a)$ as subspaces of SQ. More precisely, let

$$o(1) = \{x \in \mathrm{SQ} : x \text{ is convergent to zero}\}, \quad O(1) = \{x \in \mathrm{SQ} : x \text{ is bounded}\}$$

and for $a \in$ SQ let

$$o(a) = a\,o(1) = \{ax : x \in o(1)\}, \qquad O(a) = a\,O(1) = \{ax : x \in O(1)\}.$$

For $m \in \mathbb{N}(-1)$ let Pol (m) denote the space Ker Δ^{m+1} i.e. the space of all polynomial sequences of degree at most m.

We say that a subset U of a metric space X is a uniform neighborhood of a subset Y of X if there exists a positive number ε such that

$$\bigcup_{y \in Y} \mathrm{B}(y, \varepsilon) \subset U$$

where $\mathrm{B}(y, \varepsilon)$ denotes an open ball of radius ε about y.

Let $f: \mathbb{R} \to \mathbb{R}$. We say that a sequence $y \in$ SQ is uniformly f-bounded if f is bounded on a certain uniform neighborhood of the set $y(\mathbb{N}_0)$.