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# Numerical solution of nonlinear delay differential equations of fractional order in reproducing kernel Hilbert space

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ABSTRACT

In this paper, approximate solutions to a class of fractional differential equations with delay are presented by using a semi-analytical approach in Hilbert function space. Further, the uniqueness of the solution is proved in the space of real-valued continuous functions, as well as the existence of the solution is proved in Hilbert function space. We also prove convergence and perform an analysis error for the proposed approach. Sophisticated delay differential equations of fractional order are considered as test examples. Numerical results illustrate the efficiency of the proposed approach computationally.



## 1. Introduction

Nonlinear phenomena, which have different applications in various areas of science and engineering such as thermal systems, turbulence, image processing, fluid flow, mechanics, viscoelastic, and other areas of applications [1–10], can be modeled by fractional differential equations.

In mathematics, a delay differential equation is a type of differential equation in which the fractional derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. The delay fractional differential equations are appeared in modeling of various problems in engineering and sciences such as biology, economy, control and electrodynamics. For some applications of these equations we refer the interested reader to [11–16]. Some analytical and numerical methods have been developed for obtaining approximate solutions to delay differential equations. For instance we can mention the following works.

Ulsoy [17] used the concept of Lambert functions to present the analytical solution of system of homogeneous delay differential equations, Evans et al. [18] applied the Adomian decomposition method to approximate solutions for delay differential equations. Adomian et al. [19] used the Adomian decomposition method to obtain the approximate solution for the initial value problem of delay differential equations. The homotopy perturbation method to compute an approximation to the solution of the delay differential equations has been employed in [14–16].

Due to its importance in scientific fields, it is interesting to study the fractional model of delay differential equations. Hence, it is usually difficult to obtain closed-form solutions for fractional-order delay differential equations, especially for nonlinear types. This work aims to introduce a convenient and useful approach with high efficiency for solving a class of nonlinear fractional-order delay differential equations in the Hilbert function space.

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Minggen [20] introduced the Hilbert function space, $W_2^1[a, b]$ and its reproducing kernel. This useful framework has successfully been used for constructing approximate solutions to several nonlinear problems such as singular nonlinear second-order periodic boundary value problems [21], nonlinear system of second order boundary value problems [22], one-dimensional variable-coefficient Burgers equation [23], the coefficient inverse problem of differential [24], nonlinear age-structured population equation [25].

In this paper, an approximate solution of a class of nonlinear fractional-order delay differential equation is presented in Hilbert function space. The structure of this paper is organized as follows.

We introduce the basic definitions and properties of the fractional calculus theory in Section 2. In Section 3, the Hilbert function space $W_2^r[0, T]$ is introduced and the reproducing kernel is determined. In Sections 4 and 5, we present our main results concerning to our method. In Section 6, we present some computational results and test examples. We end the paper with few concluding remarks in Section 7.

## 2. Preliminaries and notations

### 2.1. Fractional calculus

In this section, we present some standard definitions and results used throughout this paper. For more details on the geometric and physical interpretation for fractional calculus see [26]. The Caputo and Riemann–Liouville fractional derivative/integral and their properties are defined as follows:

**Definition 2.1.** A real function $\zeta(\tau), \tau > 0$, is said to be in the space $C_\alpha$, $\alpha \in \Re$, if there exists a real number $p(>\alpha)$, such that $\zeta(\tau) = \tau^p \xi_1(\tau)$, where $\zeta_1(\tau) \in C[0, \infty)$, and it is said to be in the space $C_\alpha^m$, $m \in N$, if and only if $\zeta^{(m)}(\tau) \in C_\alpha$.

**Definition 2.2.** The Riemann–Liouville fractional integral operator $J^\alpha$ of order $\alpha \geq 0$, of a function $\zeta \in C_\alpha, \alpha \geq -1$, is defined as

$$\begin{cases} J_{0_+}^\alpha \zeta(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{(\alpha-1)} \zeta(s) ds, \quad (\alpha, \tau > 0), \\ J_{0_+}^0 \zeta(\tau) = \zeta(\tau), \end{cases} \tag{2.1}$$

where $\Gamma$ is the well-known Gamma function.

The properties of the operator $J_{0_+}^\alpha$ can be found in [26] and we mention only the following cases

(1) $J_{0_+}^\alpha J_{0_+}^\beta \zeta(\tau) = J_{0_+}^{\alpha+\beta} \zeta(\tau)$,
(2) $J_{0_+}^\alpha J_{0_+}^\beta \zeta(\tau) = J_{0_+}^\beta J_{0_+}^\alpha \zeta(\tau)$,
(3) $J_{0_+}^\alpha \tau^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \tau^{\alpha+\gamma}$,

where $\zeta \in C_\alpha, \alpha \geq -1, \alpha, \beta \geq 0$.

The Riemann–Liouville derivatives have certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator $D_{0_+}^\alpha$ which is proposed by Caputo [26].

**Definition 2.3.** The fractional derivative $D_{0_+}^\alpha$ of $\zeta(x) \in C_{-1}^n$ in the Caputo's sense is defined as

$$D_{0_+}^\alpha \zeta(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_0^\tau (\tau - s)^{n-\alpha-1} \zeta^{(n)}(s) ds, \quad (n - 1 < \alpha \leq n, \tau > 0). \tag{2.2}$$

where the parameter $\alpha$ is order of the derivative and is allowed to be real or even complex.

In the following we give two basic properties of the Caputo's fractional derivative:

(1) Let $\zeta \in C_{-1}^n$, then $D_{0_+}^\alpha \zeta, 0 \leq \alpha \leq n$, is well defined and $D_{0_+}^\alpha \zeta \in C_{-1}$.
(2) Let $n - 1 < \alpha \leq n$, and $\zeta \in C_\alpha^n, \alpha \geq -1$. Then

$$\begin{cases} J_{0_+}^\alpha D_{0_+}^\alpha \zeta(\tau) = \zeta(x) - \sum_{k=0}^{n-1} \zeta^{(k)}(0^+) \frac{\tau^k}{k!}, \\ D_{0_+}^\alpha J_{0_+}^\alpha \zeta(\tau) = \zeta(\tau). \end{cases} \tag{2.3}$$

**Theorem 2.4.** *Assume that the continuous functions $\zeta(\tau)$ and $\chi(\tau)$ have fractional derivatives of order $\alpha$, then the following properties hold,*

$$D_{0_+}^\alpha (\gamma \zeta(\tau) + \eta \chi(\tau)) = \gamma D_{0_+}^\alpha \zeta(\tau) + \eta D_{0_+}^\alpha \chi(\tau), \quad \gamma, \eta \in \mathcal{C}, \tag{2.4}$$

$$D_{0_+}^\alpha (\zeta(\tau) \chi(\tau)) \neq \chi(\tau) D_{0_+}^\alpha \zeta(\tau) + \zeta(\tau) D_{0_+}^\alpha \chi(\tau). \tag{2.5}$$