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Generalized coupled fixed points and its application to a class of systems of functional equations arising in dynamic programming

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ABSTRACT

In this paper, we introduce the definition of generalized coupled fixed point in the space of the bounded functions on a set S and we prove a result about the existence and uniqueness of such points. As an application of our result, we study the problem of existence and uniqueness of solutions for a class of systems of functional equations which appears in dynamic programming.

1. Introduction

The Banach contraction mapping principle is one of the pivotal results of analysis. Its significance lies in its vast applicability in a great number of branches of mathematics and other sciences.

Generalizations of the above principle have been objects of study in a lot of papers appearing in the literature. Particularly, one of these generalizations is due to Rhoades [1] and he uses weakly contractive mappings. Earlier to present the definition of this class of mappings, we introduce the class $\mathcal{A}$ of functions $\varphi : [0, \infty] \to [0, \infty]$ which is nondecreasing and $\varphi(t) = 0$ if and only if $t = 0$. Examples of functions in the class $\mathcal{A}$ are $\varphi(t) = \lambda t$ with $\lambda \in (0, 1)$, $\varphi(t) = arctgt$, $\varphi(t) = ln(1 + t)$ and $\varphi(t) = \frac{t}{1+t}$, among others.

Definition 1. Let (X, d) be a metric space and let $T : X \to X$ be a mapping. We say that T is weakly contractive if, for any $x, y \in X$,

$$d(Tx, Ty) \le d(x, y) - \varphi(d(x, y)),$$

where $\varphi \in \mathcal{A}$.

The following fixed point theorem which appears in [1] will be a crucial tool in our study.

Theorem 1 ([1]). *Let (X, d) be a complete metric space and $T : X \to X$ a weakly contractive mapping. Then T has a unique fixed point.*

Remark 1. In [1], the author assumes that $\lim_{t\to\infty} \varphi(t) = \infty$ and the continuity of φ, but a detailed analysis of the proof says us that these conditions are superfluous.

The main purpose of this paper is to introduce the definition of generalized coupled fixed point, to prove a result about the existence and uniqueness of these points and to apply the result to a problem which appears in dynamic programming. Our main tool in our study is Theorem 1.

This topic has been treated recently in some papers (see, for example [2–6]).

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2. Main result

In this section, we consider a nonempty set S and by $B(S)$ we will denote the set of all bounded real functions defined on S. According to the ordinary addition of functions and scalar multiplication, $B(S)$ is a real vectorial space on $\mathbb{R}$. In $B(S)$, we consider the classical norm

$$\|u\| = \sup_{x\in S} |u(x)|, \quad \text{for } u \in B(S),$$

and it is well known that $(B(S), \|.\|)$ is a Banach space.

Notice that the distance in $B(S)$ is given by

$$d(u, v) = \sup\{|u(x) - v(x)| : x \in S\}, \quad \text{for } u, v \in B(S).$$

Definition 2. Suppose that $G : B(S) \times B(S) \to B(S)$ and $\alpha : B(S) \to B(S)$ are two mappings. An element $(u, v) \in B(S) \times B(S)$ is called an α-coupled fixed point of G if $G(u, v) = u$ and $G(\alpha(u), \alpha(v)) = v$.

Earlier to present our main result, we needed to introduce the class of functions $\mathcal{B}$ given by those functions $\varphi : [0, \infty] \to [0, \infty]$ which are nondecreasing and such that $I - \varphi \in \mathcal{A}$, where I denotes the identity mapping on $[0, \infty]$ and $\mathcal{A}$ is the class of functions introduced in Section 1.

Examples of functions belonging to $\mathcal{B}$ are $\varphi(t) = arctgt$, $\varphi(t) = ln(1 + t)$, among others.

We are ready to present the main result of the paper which gives us sufficient condition for the existence and uniqueness of an α-coupled fixed point.

Theorem 2. *Suppose that $G : B(S) \times B(S) \to B(S)$ and $\alpha : B(S) \to B(S)$ are two mappings. Assume that G satisfies $d(G(x, y), G(u, v)) \le \varphi(max(d(x, u), d(y, v)))$, for any $x, y, u, v \in B(S)$, where $\varphi \in \mathcal{B}$, and that the mapping α is non-expansive (this means that $d(\alpha(x), \alpha(y)) \le d(x, y)$ for any $x, y \in B(S)$). Then G has a unique α-coupled fixed point.*

Proof. Consider the cartesian product $B(S) \times B(S)$ endowed with the distance

$$\bar{d}((x, y), (u, v)) = max(d(x, u), d(y, v)),$$

for any $(x, y), (u, v) \in B(S) \times B(S)$. It is known that $(B(S) \times B(S), \bar{d})$ is a complete metric space.

Now, we consider the mapping $\bar{G} : B(S) \times B(S) \to B(S) \times B(S)$ defined by

$$\bar{G}(x, y) = (G(x, y), G(\alpha(x), \alpha(y))).$$

Next, we check that $\bar{G}$ satisfies assumptions of Theorem 1, i.e., $\bar{G}$ is a weakly contractive mapping on $B(S) \times B(S)$.

In fact, taking into account our assumption, for any $x, y, u, v \in B(S)$, we have

$$\begin{aligned}\bar{d}(\bar{G}(x, y), \bar{G}(u, v)) &= \bar{d}((G(x, y), G(\alpha(x), \alpha(y))), (G(u, v), G(\alpha(u), \alpha(v))))\\ &= max\{d(G(x, y), G(u, v)), d(G(\alpha(x), \alpha(y)), G(\alpha(u), \alpha(v)))\}\\ &\le max\{\varphi(max(d(x, u), d(y, v))), \varphi(max(d(\alpha(x), \alpha(u)), d(\alpha(y), \alpha(v))))\}.\end{aligned}$$

Since the mapping α is non-expansive, $max(d(\alpha(x), \alpha(u)), d(\alpha(y), \alpha(v))) \le max(d(x, u), d(y, v))$, and, since φ is nondecreasing, we infer

$$\begin{aligned}\bar{d}(\bar{G}(x, y), \bar{G}(u, v)) &\le \varphi(max(d(x, u), d(y, v)))\\ &= max(d(x, u), d(y, v)) - (max(d(x, u), d(y, v)) - \varphi(max(d(x, u), d(y, v)))).\end{aligned}$$

Now, taking into account that $\varphi \in \mathcal{B}$ and, consequently, $I - \varphi \in \mathcal{A}$, from the last expression we obtain that $\bar{G}$ is a weakly contractive mapping. By using Theorem 1, there exists a unique $(x_0, y_0) \in B(S) \times B(S)$ such that $\bar{G}(x_0, y_0) = (x_0, y_0)$. This means that $G(x_0, y_0) = x_0$ and $G(\alpha(x_0), \alpha(y_0)) = y_0$ and, therefore, (x_0, y_0) is the unique α-coupled fixed point of G.

This finishes the proof. □

Remark 2. Notice that the same argument used in the proof of Theorem 2 works when we consider as mapping $\bar{G}$ the one defined by $\bar{G}(x, y) = (G(y, x), G(\alpha(y), \alpha(x)))$ or $\bar{G}(x, y) = (G(x, y), G(\alpha(y), \alpha(x)))$, for example, and we obtain existence and uniqueness of other class of coupled fixed points.

3. Application to dynamic programming

The following types of systems of functional equations

$$\begin{cases} u(x) = \sup_{y\in D}\{g(x, y) + F(x, y, u(T(x, y)), v(T(x, y)))\} \\ v(x) = \sup_{y\in D}\{g(x, y) + F(x, y, \alpha(u(T(x, y))), \alpha(v(T(x, y))))\} \end{cases} \tag{1}$$

appear in the study of dynamic programming (see [7]), where $x \in S$ and S is a state space, D is a decision space, $T : S \times D \to S$, $g : S \times D \to \mathbb{R}$, $\alpha : B(S) \to B(S)$ and $F : S \times D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given mappings.