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Derivatives of tangent function and tangent numbers

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ABSTRACT

In the paper, by induction, the Faà di Bruno formula, and some techniques in the theory of complex functions, the author finds explicit formulas for higher order derivatives of the tangent and cotangent functions as well as powers of the sine and cosine functions, obtains explicit formulas for two Bell polynomials of the second kind for successive derivatives of sine and cosine functions, presents curious identities for the sine function, discovers explicit formulas and recurrence relations for the tangent numbers, the Bernoulli numbers, the Genocchi numbers, special values of the Euler polynomials at zero, and special values of the Riemann zeta function at even numbers, and comments on five different forms of higher order derivatives for the tangent function and on derivative polynomials of the tangent, cotangent, secant, cosecant, hyperbolic tangent, and hyperbolic cotangent functions.

1. Main results

It is well known that the tangent function $\tan x$ can be expanded into the Maclaurin series

$$\tan x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}x^{2k-1} = \sum_{k=1}^{\infty} T_{2k-1}\frac{x^{2k-1}}{(2k-1)!}, \quad |x| < \frac{\pi}{2},$$

see [1, p. 75, 4.3.67] and [5, p. 259], where T_{2k-1} are called the tangent numbers or zag numbers and B_n for $n \ge 0$ are the Bernoulli numbers which may be defined by the power series expansion

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n\frac{x^n}{n!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{x^{2k}}{(2k)!}, \quad |x| < 2\pi.$$

The tangent numbers T_{2k-1} may also be defined combinatorially as the numbers of alternating permutations on $2k-1 = 1, 3, 5, 7, \dots$ symbols (where permutations that are the reverses of one another counted as equivalent). The first few tangent numbers T_{2k-1} for $k = 1, 2, \dots, 5$ are 1, 2, 16, 272, 7936.

It is clear that

$$T_{2k-1} = (-1)^{k-1}\frac{2^{2k-1}(2^{2k}-1)}{k}B_{2k} = \lim_{x\to 0}\tan^{(2k-1)} x. \tag{1.1}$$

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Consequently, one way to compute the tangent numbers T_{2k-1} and the Bernoulli numbers B_{2k} is to find explicit formulas for $\tan^{(2k-1)} x$.

It is well known that the Riemann zeta function $\zeta(s)$ may be defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \Re(s) > 0,$$

that the Genocchi numbers G_k are given by the generating function

$$\frac{2z}{e^z+1} = \sum_{k=1}^{\infty} G_k \frac{z^k}{k!}, \quad |z| < \pi,$$

and that the Euler polynomials $E_k(x)$ are defined by

$$\frac{2e^{xz}}{e^z+1} = \sum_{k=0}^{\infty} E_k(x) \frac{z^k}{k!}, \quad |z| < \pi.$$

The tangent numbers T_{2k-1}, the Bernoulli numbers B_{2k}, the Genocchi numbers G_k, the Euler polynomials $E_k(x)$, and the Riemann zeta function $\zeta(s)$ have close relations.

In combinatorics, the Bell polynomials of the second kind, or say, the partial Bell polynomials, denoted by $\mathrm{B}_{n,k}$ for $n \ge k \ge 0$, are defined by

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \le q \le n, \ell_q \in \{0\} \cup \mathbb{N} \\ \sum_{q=1}^{n} i\ell_q = n \\ \sum_{q=1}^{n} \ell_q = k}} \frac{n!}{\prod_{q=1}^{n-k+1} \ell_q!} \prod_{q=1}^{n-k+1} \left(\frac{x_q}{q!}\right)^{\ell_q}.$$

See [5, p. 134, Theorem A]. In combinatorial analysis, the Faà di Bruno formula plays an important role and may be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ by

$$\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=1}^{n} f^{(k)}(h(t)) \mathrm{B}_{n,k}(h'(t), h''(t), \dots, h^{(n-k+1)}(t)). \tag{1.2}$$

See [5, p. 139, Theorem C].

In this paper, by induction, the Faà di Bruno formula, and some techniques in the theory of complex functions, we will establish general and explicit formulas for the nth derivatives of $\tan x$, $\cot x$, $\sin^k x$, and $\cos^k x$ for $k \in \mathbb{N}$, presents curious identities for the sine function, and obtain explicit formulas for two Bell polynomials of the second kind

$$\mathrm{B}_{m,k}\left(-\sin x, -\cos x, \sin x, \cos x, \dots, -\sin\left[x + (m-k)\frac{\pi}{2}\right]\right)$$

and

$$\mathrm{B}_{m,k}\left(\cos x, -\sin x, -\cos x, \sin x, \dots, -\cos\left[x + (m-k)\frac{\pi}{2}\right]\right).$$

By applying these formulas, we will also derive explicit formulas and recurrence relations for the tangent numbers T_{2n-1}, the Bernoulli numbers B_{2n}, the Genocchi numbers G_{2n}, special values $E_{2n-1}(0)$ of the Euler polynomials at 0, and special values $\zeta(2n)$ of the Riemann zeta function $\zeta(z)$ at even numbers $2n$. Finally, we will comment on five different forms of higher order derivatives for $\tan x$ and on derivative polynomials of $\tan x$, $\cot x$, $\sec x$, $\csc x$, $\tanh x$, and $\coth x$.

Our main results may be stated as the following theorems.

Theorem 1.1. *For $n \in \mathbb{N}$, derivatives of the tangent and cotangent functions may be computed by*

$$\begin{aligned}\tan^{(n)} x = \frac{1}{\cos^{n+1} x} \Bigg\{ &\frac{1}{2}\left[1 + \frac{1+(-1)^n}{2}\right] a_{n,\frac{1+(-1)^n}{2}} \sin\left[\frac{1+(-1)^n}{2} x + \frac{1-(-1)^n}{2}\frac{\pi}{2}\right] \\ &+ \sum_{k=1}^{\frac{1}{2}[n-1-\frac{1+(-1)^n}{2}]} a_{n,2k+\frac{1+(-1)^n}{2}} \sin\left[\left(2k + \frac{1+(-1)^n}{2}\right)x + \frac{1-(-1)^n}{2}\frac{\pi}{2}\right]\Bigg\}\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}\cot^{(n)} x = \frac{(-1)^n}{\sin^{n+1} x} \Bigg\{ &\frac{1}{2}\left[1 + \frac{1+(-1)^n}{2}\right] a_{n,\frac{1+(-1)^n}{2}} \cos\left[\frac{1+(-1)^n}{2} x\right] \\ &+ \sum_{k=1}^{\frac{1}{2}[n-1-\frac{1+(-1)^n}{2}]} (-1)^k a_{n,2k+\frac{1+(-1)^n}{2}} \cos\left[\left(2k + \frac{1+(-1)^n}{2}\right)x\right]\Bigg\},\end{aligned} \tag{1.4}$$