Applied Mathematics and Computation 268 (2015) 865–872

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

All traveling wave exact solutions of the variant Boussinesq equations☆

Wenjun Yuan[a,b], Fanning Meng[a,b,*], Yong Huang[c], Yonghong Wu[d]

[a] School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, PR China
[b] Key Laboratory of Mathematics and Interdisciplinary Sciences, Guangdong Higher Education Institutes, Guangzhou University, Guangzhou 510006, PR China
[c] School of Computer Science and Education Software, Guangzhou University, Guangzhou 510006, PR China
[d] Department of Mathematics and Statistics, Curtin University of Technology, GPO Box U 1987, Perth, WA 6845, Australia

A R T I C L E I N F O

MSC:
Primary 30D35
secondary 34A05

PACS:
02.30.Jr
02.70.Wz
02.30.-f,

Keywords:
The variant Boussinesq equations
Exact solution
Meromorphic function
Elliptic function

A B S T R A C T

In this article, we employ the complex method to obtain all meromorphic solutions of complex variant Boussinesq equations (1), then find out related traveling wave exact solutions of System (vB). The idea introduced in this paper can be applied to other nonlinear evolution equations. Our results show that all rational and simply periodic solutions $w_{r,1}(kx-\lambda t), w_{r,2}(kx-\lambda t), w_{s,1}(kx-\lambda t)$ and $w_{s,2}(kx-\lambda t)$ of System (vB) are solitary wave solutions, and there exist some rational solutions $w_{r,2}(z)$ and simply periodic solutions $w_{s,2}(z)$ which are not only new but also not degenerated successively by the elliptic function solutions. We believe that this method should play an important role for finding exact solutions in the mathematical physics. We also give some computer simulations to illustrate our main results.

1. Introduction and main results

Studies of various physical structures of nonlinear evolution equations (NLEEqs) had attracted much attention in connection with the important problems that arise in scientific applications. Exact solutions of NLEEqs of mathematical physics have attracted significant interest in the literature. Over the last years, much work has been done on the construction of exact solitary wave solutions and periodic wave solutions of nonlinear physical equations. Many methods have been developed by mathematicians and physicists to find special solutions of NLEEqs, such as the inverse scattering method [1], the Darboux transformation method [2], the Hirota bilinear method [3], the Lie group method [4], the bifurcation method of dynamic systems [5–7], the sine-cosine method [8], the tanh-function method [9,10], Fan-expansion method [11], and the homogenous balance method [12]. Practically, there is no unified technique that can be employed to handle all types of nonlinear differential equations. Recently, the complex method was introduced by Yuan et al. [13–15]. It is shown that the complex method provides a powerful mathematical tool for solving great many nonlinear partial differential equations in mathematical physics.

☆ Foundation item: Thi study was supported by the NSF of China (11271090), the NSF of Guangdong Province (S2012010010121) and the special fund of Guangdong Province and Chinese Ministry of Education on integration of production, education and research (2012B091100194).

* Corresponding author. Tel.: +86 208 6055 309.
E-mail address: mfnfdbx@163.com, gzywj@tom.com (F. Meng).

http://dx.doi.org/10.1016/j.amc.2015.06.088
0096-3003/

In 1995, Wang [12] considered the variant Boussinesq equations

$$\begin{cases} u_t + (uw)_x + w_{xxx} = 0, \\ w_t + u_x + ww_x = 0. \end{cases} \tag{vB}$$

As a model for water waves, w is the velocity and u is the total depth, and the subscripts denote partial derivatives. By using his homogenous balance method, he constructed solitary wave solutions of System (vB). Up to now, many authors (cf. [16–21]) have also considered System (vB) and found their abundant traveling wave exact solutions including doubly periodic Jacobi elliptic function solutions. In the limited cases, these solutions degenerate to the corresponding solitary wave solutions, shock wave solutions or trigonometric function (simply periodic) solutions.

In order to state our main results, we need some concepts and notations.

A meromorphic function $w(z)$ means that $w(z)$ is holomorphic in the complex plane $\mathbf{C}$ except for poles. α, b, c, c_i and c_{ij} are constants, which may be different from each other in different place. We say that a meromorphic function w belongs to the class W if w is an elliptic function, or a rational function of $e^{\alpha z}$, $\alpha \in \mathbf{C}$, or a rational function of z.

Substituting the traveling wave transformation

$$u(x,t) = u(z), \quad w(x,t) = w(z), \quad z = kx + \lambda t \tag{TWT}$$

into System (vB), and integrating it yields

$$\begin{cases} \lambda u + kuw + k^3 w'' + C_1 = 0, \\ \lambda w + ku + \dfrac{k}{2} w^2 + C_2 = 0, \end{cases} \tag{1}$$

where C_1 and C_2 are constants.

Solving Eqs. (1), we get the relation

$$u = -\frac{1}{2} w^2 - \frac{\lambda}{k} w - C_2, \tag{2}$$

and the auxiliary ordinary differential equation

$$k^3 w'' - \frac{\lambda^2 + C_2}{k} w - \frac{3\lambda}{2} w^2 - \frac{k}{2} w^3 + C_3 = 0, \tag{3}$$

where $C_3 = C_1 - \frac{C_2 \lambda}{k}$.

In this article, we employ the complex method to obtain all meromorphic exact solutions of complex Eq. (3) at first, then combining the relation (2) with transform (TWT) to find out all traveling wave exact solutions of System (vB). The idea introduced in this paper can be applied to other nonlinear evolution equations. Our results show that there exist some rational solutions $w_{r,2}(z)$ and simply periodic solutions $w_{s,2}(z)$ which are not only new but also not degenerated successively by the elliptic function solutions. In Section 4, we give some computer simulations to illustrate our main results.

Our main results are the following Theorems 1 and 2.

Theorem 1. *Suppose that $k \neq 0$, then all meromorphic solutions w of Eq. (3) belong to the class W. Furthermore, Eq. (3) has the following three forms of solutions:*

(I) *All elliptic function solutions*

$$w_d(z) = \pm k \frac{(-\wp + C)(4\wp C^2 + 4\wp^2 C + 2\wp' D - \wp g_2 - C g_2)}{((12C^2 - g_2)\wp + 4C^3 - 3Cg_2)\wp' + 4D\wp^3 + 12DC\wp^2 - 3Dg_2\wp - DCg_2} - \frac{\lambda}{k},$$

where $C_2 = \frac{\lambda^2}{2}, C_3 = -\frac{\lambda^3}{2k^2}, g_3 = 0, D^2 = 4C^3 - g_2 C$, g_2 and C are arbitrary constants.

(II) *All simply periodic solutions*

$$w_{s,1}(z) = \pm k\alpha \coth \frac{\alpha}{2}(z - z_0) - \frac{\lambda}{k},$$

and

$$w_{s,2}(z) = \pm k\alpha \left(\coth \frac{\alpha}{2}(z - z_0) - \coth \frac{\alpha}{2}(z - z_0 - z_1) \right) - \frac{\lambda}{k} \mp k\alpha \coth \frac{\alpha}{2} z_1,$$

where $z_0 \in \mathbf{C}. C_2 = \frac{1}{2}(\lambda^2 - k^4\alpha^2), C_3 = \frac{1}{2k^2}\lambda(\lambda^2 - k^4\alpha^2)$ in the former case, or $C_2 = -\frac{k^4\alpha^2}{2}(\frac{3}{\sinh^2 \frac{\alpha}{2} z_1} + 1)$, $C_3 = \frac{1}{2k^2}(\lambda \pm k^2\alpha \coth \frac{\alpha}{2} z_1)(\frac{2k^4\alpha^2}{\sinh^2 \frac{\alpha}{2} z_1} - \lambda^2 \pm \lambda k^2 \alpha \coth \frac{\alpha}{2} z_1), z_1 \neq 0$ in the latter case.

(III) *All rational function solutions*

$$w_{r,1}(z) = \pm \frac{2k}{z - z_0} - \frac{\lambda}{k}$$