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A general approach to the construction of nonconforming finite elements on convex polytopes

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ABSTRACT

This paper establishes a general approach for constructing a new class of nonconforming finite elements on arbitrary convex polytope. Our contributions generalize or complete several well-known nonconforming finite elements such as: the Crouzeix–Raviart triangle element, the Han parallelogram element, the nonconforming rotated parallelogram element of Rannacher and Turek, and several others.

1. Introduction

The finite element method in a generalized form leads us to the following situation:

We have a convex polytope $K \subset \mathbb{R}^d$ with facets $F_1, \ldots, F_n$, where $n \geq d+1$. With each facet F_j there is associated a linear functional $L_j : C(K) \to \mathbb{R}$. Furthermore, we have a finite dimensional subspace $R(K)$ of $C(K)$. We want that for each $f \in C(K)$, there exists a uniquely determined $g \in R(K)$ such that

$$L_j(g) = L_j(f) \qquad (j = 1, \ldots, n). \tag{1}$$

These are n conditions. Hence, if the functionals $L_1, \ldots, L_n$ are "sufficiently different", we have to choose $R(K)$ as an n-dimensional space.

If the polytope K is a non-degenerate simplex, then $n = d+1$. In this case, it is natural and reasonable to choose $R(K)$ as the space

$$\mathcal{P}_1 := \operatorname{span}\{1, x_1, \ldots, x_d\} \tag{2}$$

of affine functions on $\mathbb{R}^d$. When $n > d+1$, we want to enrich the space $\mathcal{P}_1$ and choose $R(K)$ as an extension of $\mathcal{P}_1$, that is,

$$R(K) := \operatorname{span}\{1, x_1, \ldots, x_d, f_1, \ldots, f_{n-d-1}\} \tag{3}$$

with appropriate functions $f_1, \ldots, f_{n-d-1} \in C(K)$. This raises the question as to what are *sufficiently different* functionals $L_1, \ldots, L_n$ and *appropriate* enrichment functions $f_1, \ldots, f_{n-d-1}$.

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We shall proceed as follows. In the next section, we fix our notation. In Section 3, we introduce a separation property which guarantees us that the functionals L_j are sufficiently different for the quasi-interpolation (1). We characterize this separation property by means of linear algebra and study it for some natural choices of $L_1, \ldots, L_n$.

The main result follows in Section 4, where we characterize all functions $f_1, \ldots, f_{n-d-1}$ that are admissible in (3) when the functionals $L_1, \ldots, L_n$ satisfy the separation property.

In Section 5, we specialize K to an orthogonal parallelepiped and make the previous results more concrete. For such polytopes, the extension of $\mathcal{P}_1$ to $R(K)$ has often been achieved by adding quadratic functions such as $f_i(\boldsymbol{x}) = x_i^2$ or $f_i(\boldsymbol{x}) = x_i^2 - x_{i+1}^2$; see [1–9,11–17]. More generally, we study if $f_1, \ldots, f_{n-d-1}$ in (3) can be generated from one single function $f : [0, 1] \to \mathbb{R}$.

2. Notation

When d is an integer greater than 1, we denote the elements of $\mathbb{R}^d$ by bold-faced letters. The corresponding normal letter with a subscript i is used for specifying the ith component. When we interpret $\boldsymbol{x} \in \mathbb{R}^d$ as a vector, it should be a column vector or, equivalently, the transpose of a row vector. Thus, $\boldsymbol{x} = (x_1, \ldots, x_d)^\top$. By $\langle \cdot, \cdot \rangle$, we denote the standard inner product and by $\| \cdot \|$ the euclidean norm. Other norms are specified by attaching a subscript to the norm symbol.

For a connected set $K \subset \mathbb{R}^d$, we denote by $C(K)$ the linear space comprising all continuous functions on K. The subspace of affine functions is denoted as in (2).

When $L : C(K) \to \mathbb{R}$ is a functional and $\boldsymbol{f} = (f_1, \ldots, f_k)^\top$ for $f_1, \ldots, f_k \in C(K)$, then we define

$$L(\boldsymbol{f}) := \begin{pmatrix} L(f_1) \\ \vdots \\ L(f_k) \end{pmatrix}.$$

We use $\mathrm{d}\boldsymbol{x}$ for integration in $\mathbb{R}^d$ and $\mathrm{d}\sigma$ for integration over a $(d-1)$-dimensional surface in $\mathbb{R}^d$. The measure of a measurable set F will be denoted by $|F|$. From the context it should be clear in which dimension the measure is taken.

If F is a facet of a convex polytope in $\mathbb{R}^d$, then

$$\boldsymbol{x}_* := \frac{1}{|F|} \int_F \boldsymbol{x} \, \mathrm{d}\sigma \tag{4}$$

is the *center of gravity* of F. More generally, if $L : C(K) \to \mathbb{R}$ is a linear functional and $L(1) \neq 0$, then

$$\boldsymbol{x}_* := \frac{L(\boldsymbol{x})}{L(1)} \tag{5}$$

is the *center of gravity* of L.

3. Separating functionals

Let $K \subset \mathbb{R}^d$ be a non-degenerate convex polytope with facets $F_1, \ldots, F_n$, where $n \geq d+1$. Suppose that with each facet F_j, there is associated a linear functional $L_j : C(K) \to \mathbb{R}$ such that $L_j(1) = 1$. Now we introduce the announced separation property.

Definition 3.1. We say that the functionals $L_1, \ldots, L_n$ *separate* affine functions if for any two different affine functions p and q we have $L_j(p) \neq L_j(q)$ for at least one $j \in \{1, \ldots, n\}$.

Remark 3.2. As a consequence of their linearity, the functionals $L_1, \ldots, L_n$ do *not* separate affine functions if and only if there exists an affine function $p \neq 0$ such that $L_j(p) = 0$ for $j = 1, \ldots, n$.

Next we give two characterizations for the separation property. Denote by $\boldsymbol{x}_*^j$ the center of gravity of L_j and define

$$\boldsymbol{w}^j := \begin{pmatrix} 1 \\ \boldsymbol{x}_*^j \end{pmatrix} \in \mathbb{R}^{d+1} \qquad (j = 1, \ldots, n). \tag{6}$$

These vectors will play a particularly important role in our consideration.

Then, the following characterizations hold.

Proposition 3.3. *The following statements are equivalent:*

(i) *The functionals $L_1, \ldots, L_n$ separate affine functions.*
(ii) *The centers of gravity of $L_1, \ldots, L_n$ are not contained in a hyperplane of $\mathbb{R}^d$.*
(iii) *For the vectors (6) we have* $\operatorname{span}\{\boldsymbol{w}^1, \ldots, \boldsymbol{w}^n\} = \mathbb{R}^{d+1}$.

Proof. First we show that (i) and (iii) are equivalent. Let

$$V := \operatorname{span}\{\boldsymbol{w}^1, \ldots, \boldsymbol{w}^n\}.$$