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Bounds for the Chebyshev functional and applications to the weighted integral formulae

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ABSTRACT

The aim of this paper is to provide some error estimates for the general weighted n-point quadrature formulae by using some inequalities for the Chebyshev functional. The above results are applied to obtain some new bounds for the Gauss–Chebyshev formulae of the first and the second kind.

1. Introduction

Let us suppose $f^{(r-1)}$ is a continuous function of bounded variation on $[a, b]$ for some $r \geq 1$ and let $w\colon [a, b] \to [0, \infty)$ is some probability density function, that is integrable function satisfying $\int_a^b w(t)dt = 1$.

In paper [11] authors have proved the following two weighted quadrature formulae of Euler type:

$$\int_a^b w(t)f(t)dt = \sum_{k=1}^n A_k f(x_k) + \sum_{i=1}^r \frac{(b-a)^{i-1}}{i!}\left(\int_a^b w(t)B_i\left(\frac{t-a}{b-a}\right)dt - \sum_{k=1}^n A_k B_i\left(\frac{x_k-a}{b-a}\right)\right)\left[f^{(i-1)}(b) - f^{(i-1)}(a)\right]$$
$$- \frac{(b-a)^{r-1}}{r!}\int_a^b \left(\int_a^b w(u)B_r^*\left(\frac{u-t}{b-a}\right)du - \sum_{k=1}^n A_k B_r^*\left(\frac{x_k-t}{b-a}\right)\right)df^{(r-1)}(t) \tag{1}$$

and

$$\int_a^b w(t)f(t)dt = \sum_{k=1}^n A_k f(x_k) + \sum_{i=1}^{r-1} \frac{(b-a)^{i-1}}{i!}\left(\int_a^b w(t)B_i\left(\frac{t-a}{b-a}\right)dt - \sum_{k=1}^n A_k B_i\left(\frac{x_k-a}{b-a}\right)\right)\left[f^{(i-1)}(b) - f^{(i-1)}(a)\right]$$
$$- \frac{(b-a)^{r-1}}{r!}\int_a^b \left(\int_a^b w(u)\left(B_r^*\left(\frac{u-t}{b-a}\right) - B_r\left(\frac{u-a}{b-a}\right)\right)du$$

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$$- \sum_{k=1}^{n} A_k \left(B_r^* \left(\frac{x_k - t}{b - a} \right) - B_r \left(\frac{x_k - a}{b - a} \right) \right) \right) \mathrm{d} f^{(r-1)}(t), \tag{2}$$

where $\sum_{k=1}^{n} A_k = 1$.

If we put $r = m + s$, these formulae are exact for all polynomials of degree $\leq m - 1$.

The functions $t \mapsto B_k(t)$, $k \geq 0$, $t \in \mathbf{R}$ are Bernoulli polynomials, $B_k = B_k(0)$, $k \geq 0$ Bernoulli numbers, and $t \mapsto B_k^*(t)$, $k \geq 0$ are periodic functions of period 1, related to Bernoulli polynomials as $B_k^*(t) = B_k(t)$, $\quad 0 \leq t < 1$. More about Bernoulli polynomials, Bernoulli numbers and periodic functions B_k^* can be found in [1].

For two real functions f, g: $[a, b] \to \mathbf{R}$, the Chebyshev functional [10] is defined by

$$C(f, g) = \frac{1}{b - a} \int_a^b f(t) g(t) \mathrm{d}t - \frac{1}{(b - a)^2} \int_a^b f(t) \mathrm{d}t \cdot \int_a^b g(t) \mathrm{d}t,$$

where f, g are such that $f, g, f \cdot g \in L^1[a, b]$. The symbol $L^p[a, b]$, $1 \leq p < \infty$, stands for the space of p-power integrable functions on interval $[a, b]$ equipped with the norm $\|f\|_p = (\int_a^b |f(t)|^p \mathrm{d}t)^{1/p}$. Further, $L^\infty[a, b]$ denotes the space of all essentially bounded functions on interval $[a, b]$ with the norm $\|f\|_\infty = \operatorname{ess\,sup}_{t \in [a,b]} |f(t)|$.

For two bounded integrable functions $f, g : [a, b] \to \mathbb{R}$ and α, β, γ and δ real numbers such that $\alpha \leq f(t) \leq \beta$, and $\gamma \leq g(t) \leq \delta$, for all $t \in [a, b]$, the following inequality is well known as the Grüss inequality (see [10], p. 296)

$$|C(f, g)| \leq \frac{1}{4} (\beta - \alpha)(\delta - \gamma).$$

Over the last decades many researchers have investigated inequalities related to the Chebyshev functional and their applications in Numerical analysis (see [3,4,7,10,14] and the references cited therein).

Cerone and Dragomir [5] proved the following Grüss type inequalities:

Theorem 1. *Let f, g: $[a, b] \to \mathbf{R}$ be two absolutely continuous functions on $[a, b]$ with*

$$(\cdot - a)(b - \cdot) \left(f' \right)^2, \quad (\cdot - a)(b - \cdot) \left(g' \right)^2 \in L^1[a, b],$$

then

$$\begin{aligned} |C(f, g)| &\leq \frac{1}{\sqrt{2}} [C(f, f)]^{1/2} \frac{1}{\sqrt{b - a}} \left[\int_a^b (t - a)(b - t) \left(g'(t) \right)^2 \mathrm{d}t \right]^{1/2} \\ &\leq \frac{1}{2(b - a)} \left[\int_a^b (t - a)(b - t) \left(f'(t) \right)^2 \mathrm{d}t \right]^{1/2} \cdot \left[\int_a^b (t - a)(b - t) \left(g'(t) \right)^2 \mathrm{d}t \right]^{1/2}. \end{aligned} \tag{3}$$

The constants $1/\sqrt{2}$ and $1/2$ are the best possible.

Theorem 2. *Assume that g: $[a, b] \to \mathbf{R}$ is monotonic nondecreasing on $[a, b]$ and f: $[a, b] \to \mathbf{R}$ is absolutely continuous with $f' \in L^\infty[a, b]$, then*

$$|C(f, g)| \leq \frac{1}{2(b - a)} \|f'\|_\infty \cdot \int_a^b (t - a)(b - t) \mathrm{d}g(t). \tag{4}$$

The constant $1/2$ is the best possible.

In this paper we obtain some new estimations of the reminder in the generalized weighted quadrature formulae (1) and (2) by using Theorems 1 and 2. On this way we generalized results from papers [8,9,13]. As special cases, some error estimates for the Gauss–Chebyshev formulae of the first and the second kind are derived.

More about quadrature formulae and error estimations (from the point of view of inequality theory) can be found in monographs [2] and [6].

2. Main results

We introduce the following notation

$$G_r(t) = \int_a^b w(u) B_r^* \left(\frac{u - t}{b - a} \right) \mathrm{d}u - \sum_{k=1}^{n} A_k B_r^* \left(\frac{x_k - t}{b - a} \right),$$

and

$$F_r(t) = \int_a^b w(u) \left(B_r^* \left(\frac{u - t}{b - a} \right) - B_r \left(\frac{u - a}{b - a} \right) \right) \mathrm{d}u - \sum_{k=1}^{n} A_k \left(B_r^* \left(\frac{x_k - t}{b - a} \right) - B_r \left(\frac{x_k - a}{b - a} \right) \right). \tag{5}$$