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On the spectra of some combinations of two generalized quadratic matrices

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ABSTRACT

Let A and B be two generalized quadratic matrices with respect to idempotent matrices P and Q, respectively, such that $(A-\alpha P)(A-\beta P)=\mathbf{0}$, $AP=PA=A$, $(B-\gamma Q)(B-\delta Q)=\mathbf{0}$, $BQ=QB=B$, $PQ=QP$, $AB\neq BA$, and $(A+B)(\alpha\beta P-\gamma\delta Q)=(\alpha\beta P-\gamma\delta Q)(A+B)$ with $\alpha,\beta,\gamma,\delta\in\mathbb{C}$. Let $A+B$ be diagonalizable. The relations between the spectrum of the matrix $A+B$ and the spectra of some matrices produced from A and B are considered. Moreover, some results on the spectrum of the matrix $A+B$ are obtained when $A+B$ is not diagonalizable. Finally, some results and examples illustrating the applications of the results in the work are given.

1. Introduction and notations

Let $\mathbb{C}$ be the set of all complex numbers and $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$. The symbols $\mathbb{C}_{n,m}$, $\mathbb{C}_n$, I_n, and $\mathbf{0}$ will denote the set of all $n\times m$ complex matrices, the set of all $n\times n$ complex matrices, the identity matrix (of size n), and the zero matrix of suitable size, respectively. The rank of a $A\in\mathbb{C}_{n,m}$ will be denoted by $\mathrm{rk}(A)$. For $A\in\mathbb{C}_n$, the spectrum of A will be symbolized by $\sigma(A)$.

Let $P\in\mathbb{C}_n$ be an idempotent (i.e., $P^2=P$). We say that $A\in\mathbb{C}_n$ is a *generalized quadratic matrix with respect to P* if there exist $\alpha,\beta\in\mathbb{C}$ such that

$$(A-\alpha P)(A-\beta P)=\mathbf{0},\qquad AP=PA=A. \tag{1.1}$$

The notation $\mathfrak{L}(P;\alpha,\beta)$ will indicate the set of matrices A satisfying (1.1). From (1.1), we get the equality

$$A^2=(\alpha+\beta)A-\alpha\beta P.$$

Taking $P=I_n$ in (1.1), we get that the matrix A is an $\{\alpha,\beta\}$-quadratic matrix. Therefore, the results which will be obtained in this work are more general than the results provided in [6].

The set of $\{\alpha,\beta\}$-quadratic matrices has been extensively studied by many authors. For example, in [10] Wang obtained many results related to sums and products of two quadratic matrices. Also, the author characterized when a complex matrix T is the sum of an idempotent matrix and a square-zero matrix in [10]. In [11], the problem of characterizing matrices which can be expressed as a product of finitely many quadratic matrices were considered by Wang. Wang, considering that every complex $n\times n$ matrix T is a product of four quadratic matrices, showed that if T is invertible, then the number of required quadratic matrices

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can be reduced to three in [12]. In [13], Wang characterized the products of two and four invertible quadratic operators among normal operators and showed that every invertible operator is the product of six invertible quadratic operators. Aleksiejczyk and Smoktunowicz studied many properties of quadratic matrices in [1]. Later, Farebrother and Trenkler, extending the concept of quadratic matrix to generalized quadratic matrix, examined the Moore–Penrose and group inverse of matrices of that type in [3]. In [2], Deng gave explicit expressions for the Moore–Penrose inverse, the Drazin inverse, and the nonsingularity of the difference of two generalized quadratic operators. Also, Deng obtained spectral characterizations of generalized quadratic operators. In [9], Pazzis considered the problem of determining when a matrix is the sum of an idempotent and a square-zero matrix over an arbitrary field, introducing the concept of an (a, b, c, d)-quadratic sum.

In [5], the authors discussed the spectra of some matrices depending on two idempotent matrices. Later, in [6] it was extended those results to a pair of two quadratic matrices. In this work, we will obtain the generalization of some results given in [6] and give some additional results related to the subject.

These type of matrices should be of interest not only from the algebraic point of view but also from the role they play in applied sciences, for example, in the statistical theory: Let A be a generalized quadratic matrix such that $(A - \alpha P)(A - \beta P) = \mathbf{0}$ with $\alpha \neq \beta$ and $AP = PA = A$. Then, there exist two idempotents $X, Y \in \mathbb{C}_n$ such that $A = \alpha X + \beta Y$, $X + Y = P$, and $XY = YX = \mathbf{0}$ (Theorem 1.1, [7]). If the matrices X and Y are also real symmetric, then the matrix A becomes a linear combination of two disjoint real symmetric idempotent matrices. On the other hand, it is a well known fact that if C is an $n \times n$ real symmetric matrix and $\mathbf{x}$ is an $n \times 1$ real random vector having the multivariate normal distribution $N_n(0, I_n)$, then a necessary and sufficient condition for the quadratic form $\mathbf{x}'C\mathbf{x}$ to be distributed as a chi-square variable is that $C^2 = C$. Now, let $\mathbf{x}$ be an $n \times 1$ real random vector mentioned above. Then, the quadratic form $\mathbf{x}'A\mathbf{x}$ is a random variable distributed as a linear combination of two independent chi-square distributions.

2. Results

In this section, first it is given a theorem which examines the spectrum of a sum of generalized quadratic matrices A and B with $AB = BA$. Later, it is presented a lemma which helps to establish a relation between the spectrum of the sum of these matrices and the spectra of various combinations of these matrices in the case $AB \neq BA$.

As is easy to see, one has $A \in \mathfrak{L}(P; \alpha, \beta)$ if and only if $aA \in \mathfrak{L}(P; a\alpha, a\beta)$ for any $a \in \mathbb{C}^*$. Thus, instead of studying the spectrum of $aA + bB$ when $a, b \in \mathbb{C}^*$ and A and B are generalized quadratic, we will study the spectrum of $A + B$.

We shall use the following notation for the sake of simplicity: If $\Gamma_1, \Gamma_2 \subset \mathbb{C}$, then we denote $\Gamma_1 + \Gamma_2 = \{z_1 + z_2 : z_1 \in \Gamma_1, z_2 \in \Gamma_2\}$. Note that, in general, $\Gamma + \Gamma \neq 2\Gamma$.

Theorem 2.1. *Let $A, B \in \mathbb{C}_n$ be two generalized quadratic matrices and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $A \in \mathfrak{L}(P, \alpha, \beta)$, $B \in \mathfrak{L}(Q, \gamma, \delta)$. If $AB = BA$, then $\sigma(A + B) \subset \{0, \alpha, \beta\} + \{0, \gamma, \delta\}$.*

Proof. Since P is an idempotent matrix, there exists a nonsingular matrix $S \in \mathbb{C}_n$ such that $P = S(I_r \oplus \mathbf{0})S^{-1}$ with $r = \mathrm{rk}(P)$. From $AP = PA = A$, we get that A can be written as $A = S(X \oplus \mathbf{0})S^{-1}$ where $X \in \mathbb{C}_r$. Also, we have $(X - \alpha I_r)(X - \beta I_r) = \mathbf{0}$ since $(A - \alpha P)(A - \beta P) = \mathbf{0}$. From $X^2 - (\alpha + \beta)X + \alpha\beta I_r = \mathbf{0}$, we have that if $\lambda \in \sigma(X)$, then $\lambda^2 - (\alpha + \beta)\lambda + \alpha\beta = 0$, and therefore, $\lambda \in \{\alpha, \beta\}$. From $A = S(X \oplus \mathbf{0})S^{-1}$, we get that $\sigma(A) \subset \{0\} \cup \sigma(X) \subset \{0, \alpha, \beta\}$. In the same way, we get $\sigma(B) \subset \{0, \gamma, \delta\}$. Thus, applying Theorem 2.4.9 of [4] to the matrices A and B, we get the desired result. □

Lemma 2.1. *Let $A, B, P, Q \in \mathbb{C}_n$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $A \in \mathfrak{L}(P; \alpha, \beta)$, $B \in \mathfrak{L}(Q; \gamma, \delta)$, $(\alpha\beta P - \gamma\delta Q)(A + B) = (A + B)(\alpha\beta P - \gamma\delta Q)$, and $AB \neq BA$. Let $A + B$ be diagonalizable. Then the following statements are true:*

(i) *There exist a nonsingular $S \in \mathbb{C}_n$ and $A_0, \ldots, A_k$, $B_0, \ldots, B_k$, $P_0, \ldots, P_k$, $Q_0, \ldots, Q_k$ such that $A_i, B_i, P_i, Q_i \in \mathbb{C}_{m_i}$, $A_i \in \mathfrak{L}(P_i; \alpha, \beta)$, $B_i \in \mathfrak{L}(Q_i; \gamma, \delta)$, for $i = 0, \ldots, k$,*

$$A = S((\oplus_{i=1}^{k} A_i) \oplus A_0)S^{-1}, \qquad B = S((\oplus_{i=1}^{k} B_i) \oplus B_0)S^{-1},$$

$$P = S((\oplus_{i=1}^{k} P_i) \oplus P_0)S^{-1}, \qquad Q = S((\oplus_{i=1}^{k} Q_i) \oplus Q_0)S^{-1},$$

$A_0B_0 = B_0A_0$, $P_0Q_0 = Q_0P_0$, $A_iB_i \neq B_iA_i$ for $i = 1, \ldots, k$.

(ii) *There exist distinct complex numbers $\mu_1, \nu_1, \ldots, \mu_k, \nu_k$ such that*

$$\alpha + \beta + \gamma + \delta = \mu_i + \nu_i, \quad \sigma(A_i + B_i) = \{\mu_i, \nu_i\},$$

$$A_iB_i + B_iA_i + \mu_i\nu_i I_{m_i} = (\gamma + \delta)A_i + (\alpha + \beta)B_i + \alpha\beta P_i + \gamma\delta Q_i$$

for $i = 1, \ldots, k$.

(iii) *If $\alpha \neq \beta$ and $PQ = QP$, then there exist nonsingular matrices S_i such that*

$$A_i = S_i\begin{bmatrix} \alpha I_{x_i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \beta I_{y_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}S_i^{-1}, \qquad P_i = S_i\begin{bmatrix} I_{x_i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{y_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}S_i^{-1},$$