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Szász–Durrmeyer type operators based on Charlier polynomials

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Dedicated to the 66th Happy Anniversary of Prof. Dr. Heiner Gonska.

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ABSTRACT

In the present paper, we introduce a Durrmeyer type modification of Szász operators based on Charlier polynomials and study a Voronovskaja type asymptotic theorem, local approximation theorem, weighted approximation, statistical convergence and approximation of functions with derivatives of bounded variation for these operators.

1. Introduction

In [5], Bernstein introduced the following linear positive operators

$$B_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right),\quad x\in[0,1],$$

and he proved that if $f\in C[0,1]$ then $B_n(f;x)$ converges uniformly to $f(x)$ in $[0,1]$.

Charlier polynomials [19] have the generating functions of the form

$$e^t\left(1-\frac{t}{a}\right)^u=\sum_{k=0}^{\infty}C_k^{(a)}(u)\frac{t^k}{k!},\quad |t|<a,\tag{1.1}$$

where $C_k^{(a)}(u)=\sum_{r=0}^{k}\binom{k}{r}(-u)_r\left(\frac{1}{a}\right)^r$ and $(m)_0=1,\ (m)_j=m(m+1)\cdots(m+j-1)$ for $j\geq 1$.

Varma and Taşdelen [28] considered the following positive linear operators involving Charlier polynomials

$$L_n(f;x,a)=e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}f\left(\frac{k}{n}\right),\tag{1.2}$$

where $a>1$ and $x\in[0,\infty)$. For the special case, $a\to\infty$ and $x-\frac{1}{n}$ instead of x, these operators reduce to the well-known Szász operators [27]. They studied some convergence properties and order of approximation for these operators.

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Gupta et al. [15] introduced the Durrmeyer type modification of Szász-beta operators and studied an asymptotic approximation and error estimation in simultaneous approximation. In 2011, Erencin [9] introduced the Durrmeyer type modification of generalized Baskakov operators and obtained some direct results. Recently, Agrawal et al. [3] considered a new kind of Durrmeyer type modification of the generalized Baskakov operators having weights of Szász basis function and studied some approximation properties of these operators. Many authors have proposed the Durrmeyer type modification of different sequence of linear positive operators and studied their approximation behaviour (cf. [2,4,12–14,16,17,29]).

Inspired by the above work, we consider a Durrmeyer type modification of the operators defined by (1.2) as follows:

For $\gamma > 0$ and $f \in C_\gamma[0,\infty) := \{f \in C[0,\infty) : f(t) = O(t^\gamma), \text{ as } t \to \infty\}$, we define

$$S_{n,a}(f;x) = e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\frac{1}{B(k+1,n)}\int_0^\infty \frac{t^k}{(1+t)^{n+k+1}}f(t)dt, \tag{1.3}$$

where $n > \gamma$, $a > 1$ and $B(k+1,n)$ is the beta function defined by $B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x, y > 0$.

We note that for $f \in C_\gamma[0,\infty)$, the integral in the right hand side of (1.3) exists for all $n > \gamma$, and hence $S_{n,a}(f;x)$ is well defined. Further, clearly $S_{n,a}(f;x)$ is a linear operator and if $f \in C_\gamma[0,\infty)$ is a non-negative function then $S_{n,a}(f;x)$ is also non-negative, in view of the fact that the Charlier polynomials $C_k^{(a)}(u)$ are positive when $a > 0$ and $u \le 0$. Hence $S_{n,a}(f;x)$ is a linear positive operator.

The aim of the present paper is to establish some approximation properties of the Szász–Durrmeyer type operators based on Charlier polynomials defined by (1.3).

2. Moment estimates

Since the Charlier polynomials play substantial role in the definition of the operators given by (1.3), we mention below some examples and properties of these non-classical polynomials:

Examples: $C_0^{(a)}(u) = 1$, $C_1^{(a)}(u) = 1 - \frac{u}{a}$, $C_2^{(a)}(u) = 1 - \frac{u}{a^2}(1+2a) + \frac{u^2}{a^2}$ and $C_3^{(a)}(u) = 1 - \frac{u}{a^3}(3a^2+3a+2) + \frac{3u^2}{a^3}(a+1) - \frac{u^3}{a^3}$ etc.

Properties:

Lemma 1. *([6], Ch. VI, p. 170) For the function $C_k^{(a)}(u)$, there hold the following:*

(i) *$C_k^{(a)}(u)$ is a polynomial in u of degree k with the coefficient of u^k as $(\frac{-1}{a})^k$;*

(ii) *$C_k^{(a)}(u)$ can be expressed in terms of Laguerre polynomials $L_k^{(u-k)}(a)$ as*

$$C_k^{(a)}(u) = k!\left(\frac{-1}{a}\right)^k L_k^{(u-k)}(a), \quad \text{where} \quad L_k^{(\alpha)}(a) = \sum_{r=0}^{k}\binom{k+\alpha}{k-r}\frac{(-a)^r}{r!};$$

(iii) *$C_k^{(a)}(u)$ satisfies the recursion relation*

$$-aC_{k+1}^{(a)}(u) = (u-k-1)C_k^{(a)}(u) + kC_{k-1}^{(a)}(u), \quad k \ge 1;$$

(iv) *$C_k^{(a)}(u)$ satisfies the discrete orthogonality property*

$$\sum_{u=0}^{\infty}\omega(u)C_m^{(a)}(u)C_n^{(a)}(u) = a^n(n!)\,\delta_{mn}, \quad \textit{where} \quad \omega(u) = \frac{e^{-a}a^u}{u!} \quad \textit{and} \quad \delta_{mn} \quad \textit{is the Kronecker delta.}$$

Lemma 2. *For $f(t) = t^i$, $i = 3, 4$, we have*

(i) $L_n(t^3;x,a) = x^3 + \frac{x^2}{n}(6 + \frac{3}{a-1}) + \frac{2x}{n^2}(\frac{1}{(a-1)^2} + \frac{3}{a-1} + 5) + \frac{5}{n^3}$;

(ii) $L_n(t^4;x,a) = x^4 + \frac{x^3}{n}(10 + \frac{6}{a-1}) + \frac{x^2}{n^2}(31 + \frac{30}{a-1} + \frac{11}{(a-1)^2}) + \frac{x}{n^3}(37 + \frac{31}{a-1} + \frac{20}{(a-1)^2} + \frac{6}{(a-1)^3}) + \frac{15}{n^4}$.

Proof. We have

$$\begin{aligned} L_n(t^3;x,a) &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\frac{k^3}{n^3} \\ &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\frac{1}{n^3}\left(\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}k(k-1)(k-2) + 3\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}k^2\right. \\ &\quad \left. -2\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}k\right) \\ &= I_1 + I_2 + I_3, \quad \text{say} \end{aligned} \tag{2.1}$$