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Estimation of unknown function of a class of integral inequalities and applications in fractional integral equations

Wu-Sheng Wang*

School of Mathematics and Statistics, Hechi University, Guangxi, Yizhou 546300, PR China

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ABSTRACT

In this paper, we establish a class of iterated integral inequalities, which includes a nonconstant term outside the integrals. The upper bound of the embedded unknown function is estimated explicitly by adopting novel analysis techniques, such as: change of variable, amplification method, differential and integration. The derived result can be applied in the study of qualitative properties of solutions of fractional integral equations.

1. Introduction

It is well known that differential equations and integral equations are important tools to discuss the rule of natural phenomena. In the study of the existence, uniqueness, boundedness, stability, oscillation and other qualitative properties of solutions of differential equations and integral equations, one often deals with certain integral inequalities. One of the best known and widely used inequalities in the study of nonlinear differential equations is Gronwall–Bellman inequality [1,2], which can be stated as follows: If u and f are non-negative continuous functions on an interval $[a, b]$ satisfying

$$u(t) \le c + \int_a^t f(s)u(s)ds, \quad t \in [a, b], \tag{1}$$

for some constant $c \ge 0$, then

$$u(t) \le c\exp\left(\int_a^t f(s)ds\right), \quad t \in [a, b].$$

Pachpatte in [5] investigated the retarded inequality

$$u(t) \le k + \int_a^t g(s)u(s)ds + \int_a^{\alpha(t)} h(s)u(s)ds, \tag{2}$$

where k is a constant. Replacing k by a nondecreasing continuous function $f(t)$ in (1), Rashid in [12] studied the following retarded inequality

$$u(t) \le f(t) + \int_a^t g(s)u(s)ds + \int_a^{\alpha(t)} h(s)u(s)ds. \tag{3}$$

* Tel.: +861 8677857348.
E-mail address: wang4896@126.com, wang4896@sina.com

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In 2011, Abdeldaim and Yakout [10] studied some new integral inequalities

$$u(t) \le u_0 + \int_0^t g(s)u(s)\Big[u(s) + \int_0^s h(\tau)\Big[u(\tau) + \int_0^\tau r(\xi)u(\xi)d\xi\Big]d\tau\Big]ds, \tag{4}$$

$$u(t) \le u_0 + \int_0^t [g(s)u(s) + q(s)]ds + \int_0^t g(s)u(s)\Big[u(s) + \int_0^s h(\tau)u(\tau)d\tau\Big]ds, \tag{5}$$

$$u^{p+1}(t) \le u_0 + \Big[\int_0^t f(s)u^p(s)ds\Big]^2 + 2\int_0^t f(s)u^p(s)\Big[u(s) + \int_0^s f(\tau)u^p(\tau)d\tau\Big]ds. \tag{6}$$

In 2014, El-Owaidy et al. [13] investigated some new retarded nonlinear integral inequalities

$$u(t) \le f(t) + \int_a^t g(s)u^p(s)ds + \int_a^{\alpha(t)} h(s)u^p(s)ds, \tag{7}$$

$$u^p(t) \le f^p(t) + \int_a^{\alpha_1(t)} g(s)u(s)ds + \int_a^{\alpha_2(t)} h(s)u(s)ds. \tag{8}$$

In 2014, Zheng [16] discussed the inequalities of the following form

$$u(t) \le C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g(s)u(s)ds, \tag{9}$$

$$u(t) \le C + \int_0^t h(s)u^p(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)u^q(s) + \int_0^T h(s)u^p(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g(s)u^q(s)ds. \tag{10}$$

During the past few years, some investigators have established a lot of useful and interesting integral inequalities in order to achieve various goals; see [3–20] and the references cited therein.

In this paper, based on the works of [10,13,16], we discuss some new integral inequalities with weak singularity

$$\begin{aligned} u(t) \le{}& f(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)u(s) \\ &\times\Big[u(s) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}h(\tau)\Big[u(\tau) + \frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-\xi)^{\alpha-1}q(\xi)u(\xi)d\xi\Big]d\tau\Big]ds, \end{aligned} \tag{11}$$

$$\begin{aligned} u(t) \le{}& f(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[g(s)u(s) + q(s)]ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)u(s)\Big[u(s) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}h(\tau)u(\tau)d\tau\Big]ds, \end{aligned} \tag{12}$$

$$\begin{aligned} u^{p+1}(t) \le{}& f^{p+1}(t) + \Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)f^{(1-p)/2}(s)u^p(s)ds\Big]^2 \\ &+ 2\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)u^p(s)\Big[u(s) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}g(\tau)f^{1-p}(\tau)u^p(\tau)d\tau\Big]ds. \end{aligned} \tag{13}$$

2. Main result

Throughout this paper, let $\mathbf{R}_+ = (0, +\infty)$, $I = [0, +\infty)$.

The modified Riemann– Liouville fractional derivative, presented by Jumarie in [17,18] is defined by the following expression.

Definition 1. The modified Riemann– Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\displaystyle\int_0^t (t-\xi)^{-\alpha}(f(\xi) - f(0))d\xi, \ 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha-n)}, \ n < \alpha < n+1, n \ge 1. \end{cases} \tag{14}$$

Definition 2. The Riemann– Liouville fractional integral of order α on the interval I is defined by

$$I_t^\alpha f(t) = \frac{1}{\Gamma(1+\alpha)}\int_0^t f(s)(ds)^\alpha = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds. \tag{15}$$

In 2014, Zheng [16] proved the property

Lemma 1. *Suppose that* $0 < \alpha < 1$, f *is a continuous function, then*

$$D_t^\alpha(I_t^\alpha f(t)) = f(t). \tag{16}$$