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Mean-field backward stochastic differential equations in general probability spaces ☆

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ABSTRACT

In this paper, we deal with a class of mean-field backward stochastic differential equations in continuous time with an arbitrary filtered probability space. We prove the existence and uniqueness of a solution for those equations with strengthened Lipschitz assumption. A comparison theorem is also established.

1. Introduction

The general (nonlinear) backward stochastic differential equations (BSDEs) were firstly introduced by Pardoux and Peng [27] in 1990. Since then, BSDEs have been studied with great interest, and they have gradually become an important mathematical tool in many fields such as, financial mathematics, stochastic games and optimal control, etc, see for example, Peng [28], Hamadène and Lepeltier [18] and El Karoui et al. [17].

McKean–Vlasov stochastic differential equation of the form

$$dX(t) = b(X(t), \mu(t))dt + dW(t), \quad t \in [0, T], \quad X(0) = x,$$

where

$$b(X(t), \mu(t)) = \int_\Omega b(X(t,\omega), X(t;\omega'))P(d\omega') = E[b(\xi, X(t))]|_{\xi = X(t)},$$

b: $R^K \times R^K \to R$ being a (locally) bounded Borel measurable function and $\mu(t;\,\cdot)$ being the probability distribution of the K-dimensional unknown process $X(t)$, was suggested by Kac [19] as a stochastic toy model for the Vlasov kinetic equation of plasma and the study of which was initiated by Mckean [26]. Since then, many authors made contributions on McKean–Vlasov type SDEs and applications, see for example, Ahmed [1], Ahmed and Ding [2], Borkar and Kumar [4], Chan [7], Crisan and Xiong [15], Kotelenez and Kurtz [20] and the references therein.

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Mathematical mean-field approaches have been used in many fields, not only in physics and chemistry, but also recently in economics, finance and game theory, see for example, Lasry and Lions [21], they have studied mean-field limits for problems in economics and finance, and also for the theory of stochastic differential games. Recently, Buckdahn et al. [5] studied a special mean-field problem in a purely stochastic approach. Furthermore, Buckdahn et al. [6] deepened the investigation of mean-field BSDEs in a rather general setting, they gave the existence and uniqueness of solutions for mean-field BSDEs with Lipschitz condition on coefficients, they also established the comparison principle for these mean-field BSDEs. On the other hand, since the works [5,6] on the mean-field BSDEs, there are some efforts devoted to its generalization. Li and Luo [22] studied reflected BSDEs of mean-field type, they proved the existence and the uniqueness for reflected mean-field BSDEs. Li [23] studied reflected mean-filed BSDEs in a purely probabilistic method, and gave a probabilistic interpretation of the nonlinear and nonlocal PDEs with the obstacles. Xu [29] obtained the existence and uniqueness of solutions for mean-field backward doubly stochastic differential equations with locally monotone coefficient as well as the comparison theorem for these equations.

However, most previous contributions to BSDEs and mean-field BSDEs have been obtained in the framework of continuous time diffusion. Recently, there are some works have been done on backward stochastic differential/difference equations with finite states (see, e.g., An et al. [3], Cohen and Elliott [9–11], Lu and Ren [24]) or in general probability spaces (see, e.g., Cohen [8], Cohen et al. [12], Cohen and Elliott [13–14]).

Here, we highlight Cohen's great contribution. More precisely, Cohen and Elliott [14] studied a new kind of BSDEs of the form

$$Y_t = \xi + \int_t^T f(s, Y_{s-}, Z_s) d\mu_s - \sum_i \int_t^T Z_s^i dM_s^i, \ \ t \in [0, T], \tag{1.1}$$

using only a separability assumption on Hilbert space $L^2(\mathcal{F}_T)$ (which will be defined later), they established the existence and uniqueness as well as a comparison theorem for solutions of these BSDEs. We point out that both the martingale and driver terms in BSDE (1.1) are permitted to jump, this provide a unification of the discrete and continuous time theory of BSDEs. Furthermore, Cohen [8] deepened the investigation on this topic, where he gave a g-expectation representation for filtration consistent nonlinear expectations in general probability space. Moreover, Cohen et al. [12] established a general comparison theorem for BSDEs based on arbitrary martingales and gave its applications to the theory of nonlinear expectations.

Motivated by above works, the present paper deal with a class of mean-field BSDEs in general probability space of the form

$$Y_t = \xi + \int_t^T E'[f(s, Y'_{s-}, Z'_s, Y_{s-}, Z_s)] d\mu_s - \sum_i \int_t^T Z_s^i dM_s^i, \ \ 0 \le t \le T, \tag{1.2}$$

for details of the coefficient f, one can see Subsection 2.3. To the best of our knowledge, so far little is known about this new kind of BSDEs. This type of equation encompasses forms of mean-field BSDEs in Buckdahn et al. [5]. Our aim is to find a pair of adapted processes (Y, Z) in an appropriate space such that (1.2) hold. We also present a comparison theorem for the solutions of these BSDEs. Compared with the comparison theorem of Buckdahn et al. [5], we need not make additional hypotheses on coefficients. Therefore, we extend, in some sense, some existing results. Moreover, it should be pointed out that the approach of this paper is inspired by Cohen and Elliott [9] and Cohen et al. [12].

The paper is organized as follows. In Section 2, we introduce some notations, preliminaries and basic assumptions. Section 3 is devoted to the proof of the existence and uniqueness of the solution to mean-field BSDEs in general probability spaces. In Section 4, we give a comparison theorem for the solutions of mean-field BSDEs.

2. Notations, preliminaries and basic assumptions

In this section, we introduce some notations, preliminaries and basic assumptions.

2.1. Martingale representations

In the sequel, let $T > 0$ be a fixed terminal time. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}$, $t \in [0, T]$, satisfying the usual conditions of completeness and right-continuity. The standard Euclidean norm on R^K is denote by $\| \cdot \|$.

Definition 2.1. For any nondecreasing process of finite variation μ, the measure introduced by μ is a measure over $\Omega \times [0, T]$ given by

$$A \to E\left[\int_{[0,T]} I_A(\omega, t) d\mu\right].$$

Here, $A \in \mathcal{F} \otimes \mathfrak{B}([0, T])$, and the integral is taken pathwise in the Stieltjes sense.

The key result used in the construction of BSDEs is the martingale representation theorem. Under the assumption that the Hilbert space $L^2(\mathcal{F}_T)$ is separable, the following result is presented in [16] (see also [25]).