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Multiple solutions for asymptotically quadratic and superquadratic elliptic system of Hamiltonian type ☆

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ABSTRACT

This paper is concerned with the following nonperiodic Hamiltonian elliptic system

$$\begin{cases} -\Delta u+V(x)u=H_v(x,u,v) & x\in\mathbb{R}^N,\\ -\Delta v+V(x)v=H_u(x,u,v) & x\in\mathbb{R}^N,\\ u(x)\to 0,\ \ v(x)\to 0\ \text{ as } |x|\to\infty, \end{cases}$$

where $z=(u,v):\mathbb{R}^N\to\mathbb{R}\times\mathbb{R}$, $N\geq 3$, and the potential $V(x)$ is nonperiodic and sign-changing. By applying a generalized linking theorem for strongly indefinite functionals, we establish the existence of multiple solutions for asymptotically quadratic nonlinearity as well as the existence of infinitely many solutions for superquadratic nonlinearity.

1. Introduction and main result

The goal of this paper is to establish the existence and multiplicity of solutions to the following elliptic system

$$\begin{cases} -\Delta u+V(x)u=H_v(x,u,v) & x\in\mathbb{R}^N,\\ -\Delta v+V(x)v=H_u(x,u,v) & x\in\mathbb{R}^N,\\ u(x)\to 0,\ \ v(x)\to 0\ \text{ as } |x|\to\infty, \end{cases} \tag{ES}$$

where $z=(u,v):\mathbb{R}^N\to\mathbb{R}\times\mathbb{R}$, $N\geq 3$, $V\in C(\mathbb{R}^N,\mathbb{R})$ and $H\in C^1(\mathbb{R}^N\times\mathbb{R}^2,\mathbb{R})$. Setting

$$\mathcal{J}=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} \quad \text{and } \mathcal{A}=\mathcal{J}(-\Delta+V),$$

then (ES) can be rewritten as $\mathcal{A}z=H_z(x,z)$. In this way, (ES) can be regarded as a Hamiltonian system (see [14]).

For the case of a bounded domain, these systems were studied by a number of authors. For instance, see [6,11–14,19,22] and the references therein. The problem (ES) settled on the whole space $\mathbb{R}^N$ was considered recently in some works. Most of them

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focused on the case $V \equiv 1$, see, for instance, [1,2,4,15,23,26,32,34]. One of the main difficulties of such type problem is the lack of compactness of the Sobolev embedding. An usual way to recover compactness is choosing some space which has compact embedding property as working space, for example, the radically symmetric function space (see [4,15,23]). The second difficulty is that the negative definite space of the quadratic form appeared in the energy functional is infinitely dimensional, i.e., the energy functional is strongly indefinite which is different to the single equation case, and so the dual variational methods are involved to avoid this difficulty, see for example, [1–3,34] and their references.

Since, Kryszewski and Szulkin [21] developed a generalized linking theorem for the strongly indefinite functionals, Bartsch and Ding [5](see also [10]) gave several weaker versions, which provide another effective way to deal with such a problem. With the aid of the generalized linking theorem [5], there are some works focused on the existence and multiplicity of solutions for the general periodic problem, see [24,28,29,33,35,40–42] and their references. Additionally, compared to the periodic problem, the nonperiodic problem becomes very complicated since the energy functional has no more translation invariance. Up to now, there is only a little works concerned the nonperiodic problem by using various variational techniques. Recently, Wang et al. [32] considered the nonperiodic superquadratic problem with Ambrosetti-Rabinowitz condition and the potential $V(x)$ is a positive constant. By using some auxiliary problem related to the "limit equation" of (ES), the authors constructed linking levels of the variational functional and proved $(C)_c$-condition and obtained one solution. However, when the potential $V(x)$ is more general and sign-changing, their approach does not work since the Palais–Smale or Cerami condition isn't satisfied. An asymptotically quadratic case was considered in Zhang et al. [38], the authors used the technical condition overcame the difficulty of the lack of compactness, i.e., by imposing a control on the size of $H(x, z)$ with respect to the behavior of $V(x)$ at infinity in x. For singularly perturbed problem with subcritical and critical nonlinearities, see Ding and Lin [17]. With the aid of parameter ε ($\varepsilon > 0$ small enough), they proved that the variational functional satisfy $(C)_c$-condition. For somewhat related results for Hamiltonian system with positive potential, we refer readers to [25,31,36,37] and [39] and their references.

Motivated by the above facts, it is quite natural to ask if certain similar results can be obtain when the potential $V(x)$ is sign-changing and the system (ES) without perturbation? As we will see, the answer is affirmative. Hence, in the present paper, we focus on the case that the potential is nonperiodic and sign-changing. So, we assume that

(V_0) $V \in C(\mathbb{R}^N, \mathbb{R})$, and there exist some x such that $V(x) < 0$, and for any $h > 0$, there exists $r_0 > 0$ such that

$$meas(\{x \in \mathbb{R}^N : |x - y| \le r_0, V(x) < h\}) \to 0, \quad \text{as} |y| \to \infty,$$

where $meas(\cdot)$ denotes Lebesgue measure.

A similar assumption for Schrödinger equation was introduced in [7] (see also [8] and [9]). On the other hand, the structure of spectrum of Schrödinger operator $-\Delta + V$ is an important problem and plays a crucial role in variational setting. But the spectrum of Schrödinger operator heavily relies on property of potential $V(x)$. So there are some fundamental problems, such as when Schrödinger operator has only point spectrum? In this paper, we will give an answer to this question under the condition (V_0). One of the main ingredients of our work are two steps, one is to show that the spectrum of Schrödinger operator consists of a sequence of eigenvalues, the other is to show that the working space has compact embedding property. Moreover, without the aid of limit problem and parameter, we obtain the existence and multiplicity of solutions both for the asymptotically quadratic case and the superquadratic case.

In the following, let $A := -\Delta + V$, $\sigma(A)$, $\sigma_d(A)$ and $\sigma_e(A)$ be the spectrum of A, the discrete spectrum of A and the essential spectrum of A, respectively.

First, we handle the asymptotically quadratic case. In what follows, $\hat{H}(x, z) = \frac{1}{2}H_z(x, z)z - H(x, z)$. Suppose that

(H_0) $H \in C^1(\mathbb{R}^N \times \mathbb{R}^2, [0, \infty))$ and $H_z(x, z) = o(|z|)$ as $|z| \to 0$ uniformly in x;
(H_1) there exists a bounded function $H_\infty \in C(\mathbb{R}^N, \mathbb{R})$ such that $|H_z(x, z) - H_\infty(x)z|/|z| \to 0$ as $|z| \to \infty$ uniformly in x, and $b := \inf H_\infty(x) > \inf[(0, \infty) \cap \sigma(A)]$;
(H_2) $\hat{H}(x, z) > 0$ if $z \neq 0$, $\hat{H}(x, z) \to \infty$ as $|z| \to \infty$.

Let k be the number of eigenvalues of the operator A lying in $(0, b)$, our main result is the following:

Theorem 1.1. *Assume that (V_0), (H_0)–(H_2) are satisfied. Then (ES) has at least one nontrial solution. Moreover, if $H(x, z)$ is even in z. Then (ES) has at least k pairs of solutions.*

Next we consider the superquadratic case. Assume that

(H_3) $H(x, z) \cdot |z|^{-2} \to \infty$ as $|z| \to \infty$ uniformly in x;
(H_4) $\hat{H}(x, z) > 0$ if $z \neq 0$, and there are $\mu > \frac{N}{2}$ and $r, c_1 > 0$, such that for $|z| \ge r$, $|H_z(x, z)|^\mu \le c_1\hat{H}(x, z)|z|^\mu$.

Theorem 1.2. *Assume that (V_0), (H_0) and (H_3)-(H_4) are satisfied. Then (ES) has at least one nontrial solution. Moreover, if $H(x, z)$ is even inz. Then (ES) has infinitely many solutions.*

Remark 1.3. There exist some functions satisfy our conditions, for example:

*Ex*1. $H(x, z) = q(x)|z|^2(1 - \frac{1}{\ln(e+|z|)})$, provided that $\inf_{x\in\mathbb{R}^N} q(x) > \inf[\sigma(A) \cap (0, +\infty)]$

and

*Ex*2. $H(x, z) = q(x)((|z|^2 - 1)\ln(1 + |z|) + \frac{|z|}{2}(2 - |z|))$, where $q(x) > 0$.