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Analysis on exponential stability of hybrid pantograph stochastic differential equations with highly nonlinear coefficients

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ABSTRACT

This paper discusses exponential stability of solutions for highly nonlinear hybrid pantograph stochastic differential equations (PSDEs). Two criteria are proposed to guarantee exponential stability of the solution. The first criterion is a Khasminskii-type condition involving general Lyapunov functions. The second is developed on coefficients of the equation in virtue of M-matrix techniques. Based on the second criterion, robust stability of a perturbed hybrid PSDE is also investigated. The theory shows how much an exponentially stable hybrid PSDE can tolerate to remain stable.

1. Introduction

Stochastic differential delay equations (SDDEs) are widely used to model those systems dependent on the present and past states(see, e.g. [1–8]). When these systems experience abrupt changes in their structures and parameters, continuous-time Markov chains are introduced to form SDDEs with Markovian switching, also known as hybrid SDDEs.

One of the important issues in the study of hybrid SDDEs is the automatic control, with current emphasis placed on asymptotic stability and boundedness arising from automatic control. There is an intensive literature in this area and we mention, for example, [9–13]. In particular, [9] and [11] are two of most cited papers while [12] is the first book in this area. In most of the above mentioned references, coefficients of those systems are assumed to satisfy local Lipschitz condition and linear growth condition. However, the linear growth condition is usually violated in many practical applications. There have been some papers discussing existence, uniqueness and stability of solutions of highly nonlinear SDDEs, for example, [14–17]. Recently, [18] discussed asymptotic stability and boundedness of solutions to nonlinear hybrid SDDEs with constant delays or differentiable bounded variable delays. Also in [19], robust exponential stability and boundedness of highly nonlinear hybrid SDDEs with constant delays were investigated.

Hybrid pantograph stochastic differential equations(PSDEs) are special SDDEs that have unbounded delays (see e.g. [20,21]). PSDEs have been frequently applied in many practical areas, such as mechanic, biology, engineering and finance. The existence-uniqueness theorem of the solution for a linear PSDE was established in [20]. On stability of a PSDE, [1] investigated the growth and decay rates of special scalar PSDEs, where equations had linear drifts with unbounded delays and diffusions without delays. [18]

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proposed a Khasminskii-type condition for a nonlinear hybrid PSDE, under which the polynomial stability of the solution could be derived. [22] extended the condition of [18] to the case that different types of functions or polynomials with different orders occurred in the Lyapunov operator. [23] investigated the exponential stability of a class of hybrid PSDE, where the coefficients were dominated by polynomials with high orders. Almost sure exponential stability of both exact and numerical solutions could be derived under such conditions. But we argue that the criteria proposed in [23] were independent on the transition matrix, so that the system would be stable at any mode. This paper will apply the technique in [19] to get exponential stability of a PSDE under suitable conditions. Compared to [23], in our result, the transition matrix of Markovian switching will play an important rule in the criterion. Also M-matrix techniques will be used to form an efficient criterion. We will show exponential stability in the pth moment and almost sure exponential stability under the same condition.

When studying asymptotic properties, robust analyses on stability and boundedness have received a great deal of attention. On SDDEs, [24] and [25] discussed robust stability of linear delay equations. [26] studied robust stochastic stability of a linear system. In [27], robust stability of uncertain linear or semilinear SDDEs had been discussed. The robust stability of a stochastic delay interval system with Markovian switching was studied in [11]. Recently in [19], the robust stability and boundedness of hybrid SDDEs with constant delay and high nonlinearity had been well treated. In this paper, after giving an efficient criterion to evaluate the exponential stability of PSDEs, robust analysis on exponential stability will also be discussed. Applying the theory, we can discuss how much the perturbation can be in order for a perturbed system remaining stable.

This article is arranged as follows. A general criterion including Lyapunov functions is proposed in Section 2, under which the PSDE system will be asymptotically bounded or exponentially stable. In Section 3, an efficient criterion with the aid of M-matrices will be discussed. Robust analyses on boundedness and exponential stability are developed in Section 4. Some examples are discussed to illustrate the theory in Section 5 and conclusions are made in Section 6.

2. General results

Throughout this paper, we use following notations. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous with $\mathcal{F}_0$ containing all P-null sets). Let $B(t) = (B_1(t), \ldots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^TA)}$. Let $\mathbb{R}_+ = [0, \infty)$.

Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j \end{cases}$$

with $\Delta > 0$. $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\Sigma_{j\neq i}\gamma_{ij}$. Assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

Denote by $C(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ the family of continuous functions from $\mathbb{R}^n \times \mathbb{R}_+$ to $\mathbb{R}_+$, also by $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ the family of continuous functions $V(x, t, i)$ from $\mathbb{R}^n \times \mathbb{R}_+ \times S$ to $\mathbb{R}_+$, such that for each $i \in S$, $V(x, t, i)$ is continuously twice differentiable in x and once in t.

Consider a hybrid pantograph stochastic differential equation

$$dx(t) = f(x(t), x(\theta t), t, r(t))dt + g(x(t), x(\theta t), t, r(t))dB(t), \tag{1}$$

with $0 < \theta < 1$. Due to its special feature, we only need to know the initial data

$$x(0) = x_0 \in \mathbb{R}^n \quad \text{and} \quad r(0) = i_0 \in S \tag{2}$$

in order to solve the equation.

The well-known conditions imposed for the existence and uniqueness of the global solution are the local Lipschitz condition and the linear growth condition (see e.g. [4]–[8]). Let us state the local Lipschitz condition.

Assumption 2.1. *For each integer $h \geq 1$, there exists a constant $K_h > 0$ such that*

$$|f(x, y, t, i) - f(\bar{x}, \bar{y}, t, i)| \vee |g(x, y, t, i) - g(\bar{x}, \bar{y}, t, i)| \leq K_h(|x - \bar{x}| + |y - \bar{y}|)$$

holds for those $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ with $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \leq h$ and any $(t, i) \in \mathbb{R}_+ \times S$.

However we will replace the linear growth condition by a more general condition, a Khasminskii-type condition as applied in [18,19,22,23], to guarantee the existence of a unique global solution. Also this condition will lead to exponential stability of the solution. Before stating the general condition, we give one more notation. Given $V(x, t, r(t)) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S, \mathbb{R}_+)$, we define the function $LV : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}$ by

$$LV(x, y, t, i) = V_t(x, t, i) + V_x(x, t, i)f(x, y, t, i) + \frac{1}{2}\mathrm{trace}[g^T(x, y, t, i)V_{xx}(x, t, i)g(x, y, t, i)] + \sum_{j=1}^{N}\gamma_{ij}V(x, t, j), \tag{3}$$