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Constant sign solutions of two-point fourth order problems ☆

Alberto Cabada*, Carlos Fernández-Gómez

Department of Mathematical Analysis, Faculty of Mathematics, University of Santiago de Compostela, Santiago de Compostela, Spain

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ABSTRACT

In this paper we characterize the sign of the Green's function related to the fourth order linear operator $u^{(4)} + M\,u$ coupled with the two point boundary conditions $u(1) = u(0) = u'(0) = u''(0) = 0$. We obtain the exact values on the real parameter M for which the related Green's function is negative in $(0, 1) \times (0, 1)$. Such property is equivalent to the fact that the operator satisfies a maximum principle in the space of functions that fulfil the homogeneous boundary conditions.

When $M > 0$ the best estimate follows from spectral theory. When $M < 0$, we obtain an estimation by studying the disconjugacy properties of the solutions of the homogeneous equation $u^{(4)} + M\,u = 0$. The optimal value is attained by studying the exact expression of the Green's function. Such study allow us to ensure that there is no real parameter M for which the Green's function is positive on $(0, 1) \times (0, 1)$.

Moreover, we obtain maximum principles of this operator when the solutions verify suitable non-homogeneous boundary conditions.

We apply the obtained results, by means of the method of lower and upper solutions, to nonlinear problems coupled with these boundary conditions.

1. Introduction

Fourth order boundary value problems represents a fundamental tool when describing elastic deflections of bridges [9].

In this paper we describe the set of the real parameters M for which the fourth order boundary value problem

$$L_M\,u(t) \equiv u^{(4)}(t) + M\,u(t) = \sigma(t), \quad \text{a.e. } t \in [0, 1]; \quad u(1) = u(0) = u'(0) = u''(0) = 0, \tag{1}$$

has nonpositive solutions, for any nonnegative L^1-function σ.

After an immediate change of variables, we obtain analogous results for the adjoint boundary conditions $u(0) = u(1) = u'(1) = u''(1) = 0$. It is not difficult to verify, [2,3], that such property is equivalent to the fact that the related Green's function is negative on $(0, 1) \times (0, 1)$.

Such conditions model the deflection of beam which is incrusted and clamped at one side and supported in the other one. This study complete the one done in [7], where the clamped beam conditions $u(0) = u(1) = u'(0) = u'(1) = 0$ have been described. Moreover it continues the work done in [6] when the simply supported case $u(0) = u(1) = u''(0) = u''(1) = 0$ has been characterized.

In this case, we distinguish two situations, depending on the sign of the real parameter M. When it is positive, we deduce the best estimate by means of the spectral theory, see, for instance, [4], [13]. For negative values of M, we use the classical theory of disconjugacy [8], to obtain an estimate on M that ensures the negativeness of Green's function. To verify that the given estimation

☆ This work is partially.

* Corresponding author. Tel.: +34881813206; fax: +34881813197.
E-mail addresses: alberto.cabada@usc.es (A. Cabada), meis113@yahoo.es (C. Fernández-Gómez).

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is optimal, by means of the Mathematica Package developed in [5] (see also [4]), we calculate the exact expression of the related Green's function. This study, combined with the spectral theory, is also applied to deduce that we cannot expect positive Green's functions for any value of the real parameter M.

We also attain, in Section 4, comparison results when non homogeneous boundary conditions are considered. Section 5 is devoted to deduce existence results for nonlinear boundary value problems by means of the lower and upper solutions method.

2. Preliminaries

In this section, we introduce the definitions and main results that will be used along the paper.

Denote by $I=[0,1]$, $L^1(I)$ be the space of the Lebesgue integrable functions on I, and consider $W^{4,1}(I)$ the usual Sobolev space of C^3-functions, with u''' absolutely continuous in I.

Definition 2.1. Let $X \subset W^{4,1}(I)$ a given set. We say that the linear operator $L_M : X \longrightarrow L^1(I)$ is *inverse positive* on X if the following property is fulfilled for all $y \in X$

$$\text{If} \quad L_M\, y(t) \ge 0 \text{ a.e } t \in I \quad \text{then} \quad y(t) \ge 0 \;\text{ for all } t \in I.$$

Analogously, the linear operator L_M is *inverse negative* on X if the following property is fulfilled for all $y \in X$

$$\text{If} \quad L_M\, y(t) \ge 0 \text{ a.e } t \in I \quad \text{then} \quad y(t) \le 0 \;\text{ for all } t \in I.$$

We refer L_M as a *strongly inverse positive* (*strongly inverse negative*) operator on X if

$$L_M\, y(t) \ge 0, \quad L_M\, y(t) \not\equiv 0 \text{ a.e } t \in I \quad \text{then} \quad y(t) > 0 \;(y(t) < 0) \text{ on } (0,1).$$

In the inverse positive case we say that L_M satisfies an *(strong) anti-maximum principle* in X. In case of inverse negativeness we refer to a *(strong) maximum principle* in X.

If we consider the set

$$X_0 = \{u \in W^{4,1}(I), \quad u(1) = u(0) = u'(0) = u''(0) = 0\}.$$

We have the following properties that characterize the maximum and anti-maximum principle as the constant sign of the related Green's function. They are particular cases of Lemmas 1.6.3 and 1.6.10, and Corollaries 1.6.6 and 1.6.12 in [4]

Proposition 2.2. *If L_M is inverse negative (inverse positive) on X_0 then L_M is invertible on X_0.*

Proposition 2.3. *The operator L_M is inverse negative (inverse positive) on X_0 if and only if the Green's function g_M, related to Problem (1), is nonpositive (nonnegative) on $I \times I$.*

Having in mind the two previous results, to obtain the validity of the comparison results for operator L_M on X_0, we must study the set of parameters for which the related Green's function has constant sign on $I \times I$. In order to make this study, we consider the following condition, introduced in [4, Section 8], for a nonpositive Green's function

(N_g) Suppose that there is a continuous function $\phi(t) > 0$ for all $t \in (0,1)$ and $k_1, k_2 \in L^1(I)$, such that $k_1(s) < k_2(s) < 0$ for a.e. $s \in I$, satisfying

$$\phi(t)\, k_1(s) \le g_M(t,s) \le \phi(t)\, k_2(s), \qquad \text{for a.e. } (t,s) \in I \times I.$$

Remark 2.4. We note that if function g_M satisfies property (N_g) then operator L_M is strongly inverse negative on X_0.

Under this assumption, we introduce the following set of parameters M in which the Green's function has constant sign

$$N_L = \{M \in \mathbb{R}, \;\text{such that}\; g_M(t,s) \le 0 \;\text{ for all } (t,s) \in I \times I\} \tag{2}$$

and

$$P_L = \{M \in \mathbb{R}, \;\text{such that}\; g_M(t,s) \ge 0 \;\text{ for all } (t,s) \in I \times I\}. \tag{3}$$

As a consequence of [4, Theorems 1.8.5 and 1.8.9] we deduce the following topological property of these sets.

Proposition 2.5. *The sets N_L and P_L are (may be empty) real intervals.*

We can describe, as a consequence of [4, Lemma 1.8.25], the set N_L as follows

Proposition 2.6. *Let $\bar{M} \in \mathbb{R}$ be fixed. Suppose that operator $L_{\bar{M}}$ is invertible in X_0, its related Green's function is nonpositive on $I \times I$, it satisfies condition (N_g) and the set N_L, defined in (2), is bounded from below.*

Then $N_L = [\bar{M} - \bar{\mu}, \bar{M} - \lambda_1)$, with $\lambda_1 < 0$ the first eigenvalue of operator $L_{\bar{M}}$, and $\bar{\mu} \ge 0$ is such that $L_{\bar{M}-\bar{\mu}}$ is invertible in X_0 and the related nonpositive Green's function $g_{\bar{M}-\bar{\mu}}$ vanishes at some points of the square $I \times I$.

So, once condition (N_g) is verified, to characterize the right side of the set N_L is enough to find the first eigenvalue of operator L_M.

Moreover, we have the following result concerning both sets