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Using waveform relaxation methods to approximate neutral stochastic functional differential equation systems ☆

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ABSTRACT

This paper extends the waveform relaxation (WR) method to neutral stochastic functional differential equations. The linear convergence theory of the continuous time WR method is established in the mean square sense when the coefficients of stochastic differential equation systems satisfy the Lipschitz condition and the contractive mapping. The discrete time WR method based on the Euler scheme, which is used in an actual implementation, was also studied. It turns out that the sequence produced by this method converges linearly to the Euler approximate solution which is convergent. In addition, we prove that the preceding methods are convergent superlinearly if the neutral term does not been split. Finally, the theory is applied to a one-dimensional model problem and checked against results obtained by numerical experiments.

1. Introduction

The waveform relaxation (WR) method, also called dynamic iteration or Picard–Lindelöf iteration, is a highly parallel iterative method for numerically solving large-scale systems of ordinary differential equations (ODEs) and first proposed by Lelarasmee et al. [16] for analyzing large-scale integrated circuit. The approach is in view of the fact that there is a weak coupling between certain blocks of elements of the circuit, then the systems of ODEs can be decoupled. The different decoupled subsystems are solved independently of each other on a parallel computer in parallel, which is one of the major advantages of the WR method. Very often the circuit consists of fast and slow components and the fast components force one to use implicit integration methods. Therefore one must solve an m-dimensional nonlinear system of equations at each timepoint before advancing to the next timepoint. In the quest for improving the efficiency of numerical simulation, it was proposed to first use some continuous-time iterations (Picard–Lindelöf iterations) to "decouple" the system and then discretize the resulting subsystems. All these facts have made the WR techniques competitive with the classical approaches to the numerical solution of differential equations based, for example, on discrete variable methods such as Runge–Kutta, linear multistep or predictor–corrector method (see also [11]).

The WR method has been applied successfully to the linear problems. Nevanlinna [22] has proved superlinear convergence of dynamic iterations for linear differential systems on finite time interval. Jansen and Vandewalle [10] deal with the acceleration of the standard WR method by successive overrelaxation techniques. Sand and Burrage [24] focus on the approach to the

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parallelism and present a highly parallel method. Nichols [23] first proposed the two-stage iterative method in 1973, then it has been further studied by other authors (see [8], [26] and [30]). The WR method has been also applied extensively to more general, time-dependent coefficient problems and to nonlinear problems. It has been extended to, for example, differential-algebraic systems of neutral type [11], integro-algebraic systems and differential-algebraic systems (see [1] and [13]), systems of second kind Volterra integral equations [3], stiff nonlinear ordinary differential equations [9], functional-differential equations [31], neutral delay differential equations [27], functional differential equations of neutral type [12], Volterra type systems of neutral differential-functional equations [29]. In these references, the error estimates are derived, which imply the linear convergence of WR methods. In particular, [3] and [29] obtain the conditions of superlinear convergence for some special cases.

Stochastic differential equations (SDEs) are increasingly used nowadays to model real-world phenomena by including effects of random perturbations on the time evolution of the state of the system (see [7] and [18]). Most of the SDEs cannot be solved explicitly and many attempts have been made to develop the efficient numerical methods (see [14] and [20]). In recent years, there has been growing interesting in extending some important theorems to more general problems. For example, [2] and [17] extend the convergence theory of numerical approximate solution of stochastic ordinary differential equations (SODEs) to stochastic delay differential equations (SDDEs), [19] extends that of SDDEs to stochastic functional differential equations (SFDEs) and [28] extends that of SFDEs to neutral stochastic functional differential equations (NSFDEs).

The WR method to SDEs was first proposed by Schurz and Schneider [25] in which L^p convergence of WR methods for numerical solving of systems of SODEs is studied and the sufficient conditions for the linear L^p convergence of the method are given. However, the conditions do not hold for some linear systems even and depend on the spectral radius of the large matrixes, which is obtained hardly. [4] and [5] extend the WR methods to SDDEs, in which the convergent conditions are replaced by the Lipschitz condition and the superlinear convergence was studied. However, the superlinear convergence was obtained only for the special splitting functions and a weak convergence criterion. To our knowledge, little is as yet known about the WR methods for SFDEs and NSFDEs although it may be necessary to extend WR methods to these equations. This paper will fill the blank.

In this paper, We will prove the continuous time WR method with quite general splitting functions for NSFDEs is convergent in mean square sense under the Lipschitz conditions and the contractive mapping. The convergence theory of the discrete time WR methods, which is used in an actual implementation, will be also studied.

2. Results

Throughout this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions. Let $W = \{W(t), t \geq 0\}$, $W(t) = (W_t^1, \ldots, W_t^d)^T$, be a d-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ denote the Euclidean vector norm as well as the matrix trace norm. Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^r)$ denote the family of continuous function φ from $[-\tau, 0]$ to $\mathbb{R}^r$ with the norm $\|\varphi\| = \sup_{-\tau\leqslant\theta\leqslant 0}|\varphi(\theta)|$. Let $\mathcal{B}(C([-\tau, 0]; \mathbb{R}^r))$ denote Borel σ-algebra of $C([-\tau, 0]; \mathbb{R}^r)$ and $L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^r)$ denote the family of $(\mathcal{F}_0, \mathcal{B}(C([-\tau, 0], \mathbb{R}^r)))$-measurable $C([-\tau, 0]; \mathbb{R}^r)$-valued random variables ξ such that $E\|\xi\|^2 < \infty$(see [21]).

In this paper, we consider the following SDEs

$$\mathrm{d}[X(t) - u(X_t)] = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}W(t), \tag{2.1}$$

on $t \in [0, T]$ with initial data $X_0 = \xi$. Such equations depend on past and present values but that involve derivatives with delays as well as the function itself, so called neutral stochastic functional differential equation. Here

$$u : C([-\tau, 0]; \mathbb{R}^r) \to \mathbb{R}^r, \quad f : [0, T] \times C([-\tau, 0]; \mathbb{R}^r) \to \mathbb{R}^r,$$
$$g : [0, T] \times C([-\tau, 0]; \mathbb{R}^r) \to \mathbb{R}^{r\times d}$$

are all continuous functionals. Moreover, $X_t = \{X(t + \theta): -\tau \leq \theta \leq 0\}$, which is regarded as a $C([-\tau, 0]; \mathbb{R}^r)$-valued stochastic process, and $\xi = (\xi(t))_{-\tau\leqslant t\leqslant 0} \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^r)$. An $\mathcal{F}_t$-adapted process $X(t)$, $-\tau \leq t \leq T$ (let $\mathcal{F}_t = \mathcal{F}_0$ for $-\tau \leq t \leq 0$), is said to be a solution of equation (2.1) if it satisfies the initial condition and, moreover, for every $t \in [0, T]$,

$$X(t) - u(X_t) = X(0) - u(X_0) + \int_0^t f(s, X_s)\mathrm{d}s + \int_0^t g(s, X_s)\mathrm{d}W(s).$$

Let $U : C([-\tau, 0]; \mathbb{R}^r) \times C([-\tau, 0]; \mathbb{R}^r) \to \mathbb{R}^r$, $F : [0, T] \times C([-\tau, 0]; \mathbb{R}^r) \times C([-\tau, 0]; \mathbb{R}^r) \to \mathbb{R}^r$ and $G : [0, T] \times C([-\tau, 0]; \mathbb{R}^r) \times C([-\tau, 0]; \mathbb{R}^r) \to \mathbb{R}^{r\times d}$ denote the splitting functions for Eq. (2.1), that is

$$u(X_t) = U(X_t, X_t), f(t, X_t) = F(t, X_t, X_t), g(t, X_t) = G(t, X_t, X_t).$$

The splitting functions are chosen to attempt to decouple systems (2.1) into easily solvable independent subsystems, which may then be solved separately.

In this paper we impose the following hypotheses:

Assumption 2.1. (Lipschitz conditions) There are some constants $K_1, K_2, \bar{K}_1, \bar{K}_2 > 0$ such that

$$|F(t, \varphi_1, \varphi_2) - F(t, \psi_1, \psi_2)|^2 \leqslant K_1||\varphi_1 - \psi_1||^2 + K_2||\varphi_2 - \psi_2||^2 \tag{2.2}$$