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Numerical solution of multi-order fractional differential equations using generalized triangular function operational matrices

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ABSTRACT

Most fractional differential equations do not have closed form solutions. Development of effective numerical techniques has been an interesting research topic for decades. In this context, this paper proposes a numerical technique, for solving linear and nonlinear multi-order fractional differential equations, based on newly computed generalized triangular function operational matrices for Riemann–Liouville fractional order integral. The orthogonal triangular functions are evolved from a simple dissection of piecewise constant orthogonal block pulse functions. Theoretical error analysis is performed to estimate the upper bound of absolute error between the exact Riemann–Liouville fractional order integral and its approximation in the triangular functions domain. Numerical examples are considered for investigating the applicability and effectiveness of proposed technique to solve multi-order fractional differential equations. The results encourage the use of orthogonal TFs for analysis of real processes exhibiting fractional dynamics.

1. Introduction

Fractional calculus is the theory of differentiation and integration of arbitrary order and was originated at the time of development of the classical calculus, hence, it is as old as classical calculus but yet a fresh topic [1]. The reason why fractional calculus has been the focus of numerous pure mathematicians and applied scientists in diverse realms of science and technology is the ability of fractional derivative to explain memory effects and hereditary properties of real world processes [2–5]. Due to local nature, the integer order derivative could not describe those concepts. In order to comprehend the inherent fractional order description of fractional order models, the corresponding fractional differential equations (FDEs) need to be solved. In this regard, analytical techniques such as Laplace transform method, fractional Green's function, Mellin transform method, power series method, etc. are already developed [6]. In reality, almost all processes are nonlinear and complex in nature, therefore, these techniques are not useful for those models. For analysis of such complicated nonlinear fractional processes, efficient numerical techniques like Predictor–Corrector method, generalized Euler's method, etc. and semi analytic-numeric techniques; Adomian decomposition method, variational iteration method, fractional differential transform method, homotopy analysis method, homotopy perturbation method, etc. can be used [7–13].

The orthogonal triangular function (TF) sets developed by Deb et al. [14] are a complementary pair of piecewise linear polynomial function sets evolved from a simple dissection of block pulse function (BPF) set [14,15]. The authors have derived a

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complementary pair of operational matrices for first order integration in the TF domain and demonstrated that the TF domain technique for dynamical systems analysis is computationally more effective than the BPF domain technique. Besides system analysis, the orthogonal TFs also find applications in system identification, optimal controller design and numerical analysis of classical integral and differential equations [16–20]. Those successful applications have made us strongly believe that TFs having enough potential to be applicable in fractional order systems. To the best of our knowledge, there is no literature until date in fractional calculus that reported the use of orthogonal TFs for solving FDEs. These facts motivated us to extend the application of orthogonal TFs to solve multi-order FDEs of the form

$$D^{\alpha}y(t)=\sum_{k=1}^{r}b_kD^{\beta_k}y(t)+cF(y(t))+f(t),\quad n-1<\alpha\le n,\quad t\in[0,T] \tag{1}$$

with initial conditions; $y^{(s)}(0)=a_s, s=0,1,2,\dots,n-1$. Here $D^{\alpha}y(t)$ is the Caputo fractional derivative of order α satisfying the relation $\alpha>\beta_1>\beta_2\cdots\beta_r$, b_k and c are real constants, $f(t)$ is known function and $F(y(t))$ can be linear or nonlinear.

To accomplish our goal, we have proposed the generalized triangular function operational matrices for estimating the fractional order integral in the TF domain and based on this result, we have proposed a numerical technique for finding approximate numerical solutions of Eq. (1). The rest of the paper is prepared as follows. Useful definitions and a few properties of fractional calculus are provided in Section 2. Generation of complementary pair of TF sets from BPF set is discussed in Section 3. Section 4 presents the basic properties of orthogonal TF sets. The procedure of estimating classical and fractional integration in TF domain is explained in Section 5. An upper bound of absolute error between the exact $\alpha-$order Riemann–Liouville fractional integral and its TF estimate is computed in Section 6. In Section 7, a numerical technique based on the results of Section 5 is proposed. Section 8 implements the proposed technique on illustrative examples. Finally, the paper is concluded in Section 9.

2. Basic operators and properties of fractional calculus

This section provides two widely utilized definitions and some operational properties of fractional calculus [6].

Definition 2.1. A real function $f(t), t>0$ is said to be in the space $C_{\mu}, \mu\in R$ if there exists a real number $p(>\mu)$, such that $f(t)=t^pf_1(t)$, where $f_1(t)\in C[0,\infty)$, and it is said to be in the space C_{μ}^n if and only if $f^{(n)}\in C_{\mu}, n\in N$.

Definition 2.2. The Riemann–Liouville fractional integral of order $\alpha(>0)$ of function $f(t)\in C_{\mu}, \mu>-1$ is defined as

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau,\quad t>0 \tag{2}$$

Definition 2.3. The Riemann–Liouville fractional derivative is

$${}_0D_t^{\alpha}f(t)=D^nJ^{n-\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t(t-\tau)^{n-\alpha-1}f(\tau)d\tau,\quad t>0,\quad n-1<\alpha\le n, n\in N \tag{3}$$

Definition 2.4. The fractional derivative of function $f(t)$ in Caputo sense is defined as

$${}_0D_*^{\alpha}f(t)=J^{n-\alpha}f^{(n)}(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau,\quad t>0 \tag{4}$$

The following are the semi group and commutative properties of fractional integral and fractional derivative.
For $f(t)\in C_{\mu}, \mu>-1$ and $\alpha,\beta>0, m,n\in N$

$$J^{\alpha}J^{\beta}f(t)=J^{\beta}J^{\alpha}f(t)=J^{\alpha+\beta}f(t) \tag{5}$$

$${}_0^CD_t^{\alpha}\left({}_0D_t^mf(t)\right)={}_0D_t^m\left({}_0^CD_t^{\alpha}f(t)\right)={}_0D_t^{\alpha+m}f(t),\quad f^{(s)}(0)=0,\quad s=n,n+1,\dots,m \tag{6}$$

$${}_0D_t^{\alpha}\left({}_0D_t^mf(t)\right)={}_0D_t^m\left({}_0D_t^{\alpha}f(t)\right)={}_0D_t^{\alpha+m}f(t),\quad f^{(s)}(0)=0,\quad s=0,1,\dots,m \tag{7}$$

$$J^{\alpha}D^{\alpha}f(t)=f(t)-\sum_{k=0}^{n-1}f^{(k)}(0)\frac{t^k}{k!} \tag{8}$$

3. Triangular functions

In this section, firstly, we review block pulse functions in brief and then we introduce the method of dissecting the block pulse function set to formulate a complementary pair of orthogonal triangular function sets.