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Traveling waves in a delayed SIR epidemic model with nonlinear incidence

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ABSTRACT

We establish the existence and non-existence of traveling wave solutions for a diffusive SIR model with a general nonlinear incidence. The existence proof is shown by introducing an auxiliary system, applying Schauder's fixed point theorem and then a limiting argument. The nonexistence proof is obtained by two-sided Laplace transform when the speed is less than the critical velocity. Numerical simulations support the theoretical results. We also point out the effects of the delay and the diffusion rate of the infective individuals on the spreading speed.

1. Introduction

Epidemiological compartmental models of SIR type has played a critical role in the development of modern mathematical epidemiology. A simple deterministic SIR model for disease outbreaks considers a population split into three compartments: the susceptible individuals S, the infected individuals I, and the recovered individuals R. The model equations are

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI, \\ \dfrac{dI}{dt} = \beta SI - \gamma I, \\ \dfrac{dR}{dt} = \gamma I, \end{cases} \tag{1.1}$$

where β is the transmission coefficient, and γ is the recovery rate. Given $S(0) = S_0 > 0$, $I(0) > 0$ and $R(0) = 0$. We know from [1] that the basic reproduction number, $R_0 = \frac{\beta S_0}{\gamma}$ is a measure of the potential for disease spread in a population.

Note that in the model (1.1), it is assumed that the population are well mixed, and the transmission are instantaneous. However, due to the large mobility of people within a country or even worldwide, spatially uniform models are not sufficient to describe a disease's diffusion [2,3]. In order for a model to be more realistic, the spatial effects should be incorporated into the model. There have been many works on spatial epidemic models (see, e.g. [4–8]). These studies concentrated on the existence and non-existence of traveling wave solutions. For example, Hosono and Ilyas [5] considered the existence of traveling wave of

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the following model

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1\dfrac{\partial^2 S(x,t)}{\partial x^2} - \beta S(x,t)I(x,t), \\ \dfrac{\partial I(x,t)}{\partial t} = d_2\dfrac{\partial^2 I(x,t)}{\partial x^2} + \beta S(x,t)I(x,t) - \gamma I(x,t). \end{cases} \tag{1.2}$$

Here d_1 and d_2 are the diffusion rates for the susceptible and infective individuals, respectively. They proved that if the basic reproduction number $\beta S_{-\infty}/\gamma > 1$, then for each $c \geqslant c^* = 2\sqrt{d_2(\beta S_{-\infty} - \gamma)}$ system (1.2) admits a traveling wave solution $(S(x+ct), I(x+ct))$ satisfying $S(-\infty) = S_{-\infty}$, $I(\pm) = 0$, $S(\infty) = S_\infty < S_{-\infty}$, while there is no traveling wave solution for (1.2) when $\beta S_{-\infty}/\gamma \leq 1$. We refer to [9–18] and their references for more related works.

In most epidemiological models, bilinear incidence rate βSI are frequently used. This works well in situation with low host population densities. However, the disease transmission process may have a nonlinear incidence rate, which has been suggested by several authors [19–22]. For example, to study the impact of the non-linearity, Korobeinikov and Maini [20] considered the incidence of the form $f(S)g(I)$ in various types of models. A more general incidence $f(S, I)$, first introduced by Feng and Thime [23,24], was also considered in many literatures [25–27].

In addition, for many infectious diseases, it is important to consider the influences of delays on the disease dynamics. In epidemiological models, delay can be caused by a variety of factors. The most notorious reasons for a delay are (i) the latency of the infection in a vector, and (ii) the latent period in a infected host [25,28]. In these cases, some time is needed before the infection in the infected host or the vector will develop to the level that is sufficient to transmit the infection further.

Incorporating these two factors into an SIR disease model with diffusion, we obtain the following model

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1\dfrac{\partial^2 S(x,t)}{\partial x^2} - f(S(x,t))g(I(x,t-\tau)), \\ \dfrac{\partial I(x,t)}{\partial t} = d_2\dfrac{\partial^2 I(x,t)}{\partial x^2} + f(S(x,t))g(I(x,t-\tau)) - \gamma I(x,t), \\ \dfrac{\partial R(x,t)}{\partial t} = d_3\dfrac{\partial^2 R}{\partial x^2} + \gamma I(x,t), \end{cases} \tag{1.3}$$

where $f(S)$ and $g(I)$ are positive and continuous functions for all $S, I > 0$ and $f(0) = g(0) = 0$, satisfying the following hypotheses:

(H1) $f'(S)$ is positive and bounded function for all $S \geq 0$;
(H2) $g'(I) > 0$, $g''(I) \leq 0$ for all $I \geqslant 0$.

It is easy to see that f is Lipschitz continuous on $[0, +\infty)$. Under the hypotheses (H1) and (H2), the forms of $f(S)g(I)$ include various types of incidence rates. If $f(S) = S$ and $g(I) = I$, then the incidence rate becomes a bilinear form, which was proposed in [29]. Moreover, if $f(S) = S$ and $g(I) = \frac{I}{1+kI}$ $(k > 0)$, then the incidence rate describes the saturated effects of the prevalence of infectious diseases [30,31].

In what follows, we shall consider the existence of traveling wave solutions of (1.3). Because R does not appear in the first two equations of (1.3), we only consider the reduced system

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1\dfrac{\partial^2 S(x,t)}{\partial x^2} - f(S(x,t))g(I(x,t-\tau)), \\ \dfrac{\partial I(x,t)}{\partial t} = d_2\dfrac{\partial^2 I(x,t)}{\partial x^2} + f(S(x,t))g(I(x,t-\tau)) - \gamma I(x,t). \end{cases} \tag{1.4}$$

Hereafter, a traveling wave solution of (1.4) is a special solution

$$S(x,t) = S(\xi), \quad I(x,t) = I(\xi), \quad \xi = x + ct \in \mathbb{R}$$

satisfying the boundary conditions

$$S(-\infty) = S_{-\infty}, \quad S(+\infty) = S_\infty < S_{-\infty}, \quad I(\pm\infty) = 0, \tag{1.5}$$

where $S_{-\infty} > 0$ is a constant representing the size of the susceptible individuals before being infected. Then for all $\xi \in \mathbb{R}$, we have

$$\begin{cases} cS'(\xi) = d_1 S''(\xi) - f(S(\xi))g(I(\xi - c\tau)), \\ cI'(\xi) = d_2 I''(\xi) + f(S(\xi))g(I(\xi - c\tau)) - \gamma I(\xi). \end{cases} \tag{1.6}$$

Our main theorems are stated as follows.

Theorem 1.1. *If $\mathcal{R}_0 := \frac{f(S_{-\infty})g'(0)}{\gamma} > 1$ and $c > c^*$, then there exists a traveling wave solution $(S(x+ct), I(x+ct))$ such that (1.5) are satisfied. Furthermore, S is non-increasing in $\mathbb{R}$, $0 \leq I(\xi) \leq S_{-\infty} - S_\infty$ for all $\xi \in \mathbb{R}$, and*

$$\int_{-\infty}^{\infty} \gamma I(x)dx = \int_{-\infty}^{\infty} f(S(x))g(I(x - c\tau))dx = c(S_{-\infty} - S_\infty).$$