Applied Mathematics and Computation 263 (2015) 233–239

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Asymptotic Formulas for Generalized Szász–Mirakyan Operators

Tuncer Acar*

Kirikkale University, Department of Mathematics, TR-71450, Yahsihan, Kirikkale, Turkey

ARTICLE INFO

MSC:
Primay 41A25
41A36

Keywords:
Szász–Mirakyan operators
Quantitative Voronovskaya theorem
Grüss–Voronovskaya-type theorem
Weighted modulus of continuity

ABSTRACT

In the present paper, we consider the general Szász–Mirakyan operators and investigate their asymptotic behaviours. We obtain quantitative Voronovskaya and quantitative Grüss type Voronovskaya theorems using the weighted modulus of continuity. The particular cases are presented for classical Szász–Mirakyan operators.

1. Introduction

In the paper [8], the authors introduced the sequence of linear Bernstein-type operators defined for $f \in C[0,1]$ by $B_n(f \circ \tau^{-1}) \circ \tau$, B_n being the classical Bernstein operators and τ being any function satisfying some certain conditions. The results obtained there showed that approximation with these new construction of Bernstein operators are sensitive and present better convergence results with the suitable selection of τ. In [6], the authors introduced a general sequence of linear Szász–Mirakyan type operators by

$$S_n^{\rho}(f;x) = e^{-n\rho(x)} \sum_{k=0}^{\infty} f\left(\rho^{-1}\left(\frac{k}{n}\right)\right) \frac{(n\rho(x))^k}{k!}, \tag{1.1}$$

where ρ is a function being continuously differentiable on $\mathbb{R}^+ := [0, \infty)$ and satisfying the conditions $\rho(0) = 0$, $\inf_{x \in \mathbb{R}^+} \rho'(x) \geq 1$. This function ρ not only characterizes the operators but also characterizes the Korovkin set $\{1, \rho, \rho^2\}$ in a weighted function space. The authors studied degree of weighted convergence and some shape-preserving properties.

The most of the results given in [6] are related to uniform convergence of the operators. Therefore, in this paper we continue to study further approximation properties of the operators (1.1). We present Voronovskaya type theorems in quantitative form. The quantitative Voronovskaya theorem and its general forms were extensively studied in [11], authors presented the quantitative Voronovskaya theorem for any positive linear operators acting on compact intervals, using the least concave majorant of the modulus of continuity. The advantage of the quantitative Voronovskaya theorems is to describe the rate of convergence and to present an upper bound for the error of approximation, simultaneously. For some other quantitative versions of Voronovskaya's theorem, we can refer the readers to [2,4,10,12,18,19]. All of mentioned theorems were given for the operators acting on compact intervals, in particular, Bernstein polynomials and their derivatives. But for the operators acting on unbounded intervals, the usual modulus of continuity does not work. Therefore, quantitative Voronovskaya theorems for general sequence of linear positive

* Corresponding author. Tel.: +9003183574242.
E-mail address: tunceracar@ymail.com

http://dx.doi.org/10.1016/j.amc.2015.04.060
0096-3003/

operators via weighted modulus of continuity studied in [3]. For quantitative Voronovskaya type theorems in weighted spaces we can refer the readers to [1,20]. We also study Grüss–Voronovskaya-type results for the operators (1.1). The application of Grüss inequality in approximation theory was given in the paper [5]. In the recent note [13], the authors applied the Grüss inequality to the operators on compact intervals and thus obtained a new approach for the Bernstein polynomials using the least concave majorant $\widetilde{\omega}$. In [9], Gal and Gonska obtained the Voronovskaya-type theorem with the aid of Grüss inequality for Bernstein operators for the first time and called it Grüss–Voronovskaya-type theorem. Here, we shall give Grüss–Voronovskaya-type theorem with the weighted modulus of continuity. As a corollary of our main results, quantitative Voronovskaya theorem for classical Szász–Mirakyan operators and Voronovskaya theorem for classical Szász–Mirakyan operators and generalized Szász–Mirakyan operators are captured. For the classical Voronovskaya theorem for Szá sz–Mirakyan operators and their modifications, one can consult the papers [7,16,17] and references therein.

2. Auxiliary results

For our main results we shall need following moments, central moments and a weighted modulus of continuity. Since the moments are similar to the corresponding results for the Szász–Mirakyan operators we give the lemmas without proofs.

Lemma 1. *We have*

$$S_n^{\rho}(1;x) = 1, \quad S_n^{\rho}(\rho;x) = \rho(x),$$
$$S_n^{\rho}\left(\rho^2;x\right) = \rho^2(x) + \frac{\rho(x)}{n},$$

Lemma 2. *If we define the central moment operator by*

$$M_{n,m}^{\rho}(x) = S_n^{\rho}\left((\rho(t)-\rho(x))^m;x\right),$$

then we have

$$M_{n,0}^{\rho}(x) = 1, \quad M_{n,1}^{\rho}(x) = 0, \tag{2.1}$$

$$M_{n,2}^{\rho}(x) = \frac{\rho(x)}{n}, \quad M_{n,3}^{\rho}(x) = \frac{\rho(x)}{n^2}, \tag{2.2}$$

$$M_{n,4}^{\rho}(x) = \frac{3\rho^2(x)}{n^2} + \frac{\rho(x)}{n^3}, \tag{2.3}$$

$$M_{n,6}^{\rho}(x) = \frac{15\rho^3(x)}{n^3} + \frac{25\rho^2(x)}{n^4} + \frac{\rho(x)}{n^5}, \tag{2.4}$$

for all $n, m \in \mathbb{N}$.

Throughout the paper we consider the weight function ρ satisfying the following assumptions:

(i) ρ is a continuously differentiable function on $\mathbb{R}^+$ and $\rho(0) = 0$.
(ii) $\inf_{x\geq 0}\rho'(x) \geq 1$.

Let $\varphi(x) = 1 + \rho^2(x)$ and $B_{\varphi}(\mathbb{R}^+) = \{f : |f(x)| \leq M_f\varphi(x)\}$, where M_f is constant which may depend only on f. $C_{\varphi}(\mathbb{R}^+)$ denotes the subspace of all continuous functions in $B_{\varphi}(\mathbb{R}^+)$. By $C_{\varphi}^{*}(\mathbb{R}^+)$, we denote the subspace of all functions $f \in C_{\varphi}(\mathbb{R}^+)$ for which $\lim_{x\to\infty}\frac{f(x)}{\varphi(x)}$ is finite. Also let $U_{\varphi}(\mathbb{R}^+)$ be the space of functions $f \in C_{\varphi}(\mathbb{R}^+)$ such that f/φ is uniformly continuous. $B_{\varphi}(\mathbb{R}^+)$ is the linear normed space with the norm $\|f\|_{\varphi} = \sup_{x\geq 0}\frac{f(x)}{\varphi(x)}$.

For each $f \in C_{\varphi}(\mathbb{R}^+)$ and for every $\delta > 0$ the weighted modulus of continuity defined in [14] is given by

$$\omega_{\rho}(f,\delta) = \sup_{\substack{x,y\geq 0 \\ |\rho(y)-\rho(x)|\leq\delta}} \frac{|f(y)-f(x)|}{\varphi(y)+\varphi(x)}$$

and it was shown that:

Lemma 3. *([14]) For every $f \in U_{\varphi}(\mathbb{R}^+)$, $\lim_{\delta\to 0}\omega_{\rho}(f,\delta) = 0$ and*

$$|f(y)-f(x)| \leq (\varphi(y)+\varphi(x))\left(2 + \frac{|\rho(y)-\rho(x)|}{\delta}\right)\omega_{\rho}(f,\delta). \tag{2.5}$$

Remark 1. If $\rho(x) = x$, then ω_{ρ} is equivalent with Ω_2 given in [15]

$$\Omega_2(f;\delta) = \sup_{\substack{x,y\geq 0 \\ |h|\leq\delta}} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}.$$