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Numerical solution of fractional diffusion equation over a long time domain

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ABSTRACT

In this paper, we propose a method to compute approximate solutions to one dimensional fractional diffusion equation which requires solution for a long time domain. For this, we use a set of shifted Legendre polynomials for the space domain and a set of Legendre rational functions for the time domain. The unknown solution is approximated by using these sets of orthogonal functions with unknown coefficients and the fractional derivative of the approximate solution is represented by an operational matrix, resulting in a linear system with the unknown coefficients. Numerical examples are given to demonstrate the effectiveness of the method.

1. Introduction

Fractional differential equations are generalizations of integer order differential equations obtained by replacing some integer order derivatives by fractional ones. In comparison with integer order differential equations, the fractional differential equations show many advantages over the simulation of natural physical processes and dynamical systems [14]. A history of fractional differential operators can be found in [15,19]. Nowadays, there are many works on fractional calculus. For examples, fractional calculus is applied to model the nonlinear oscillations of earthquakes [10], continuum and statistical mechanics [13], signal processing [16] and control theory [4].

The class of orthogonal Legendre functions has found wide application in science and engineering. One advantage of using an orthogonal basis is reducing the problem to a system of linear or nonlinear algebraic equations [9]. This can be done by truncating series of orthogonal basis functions for the solution of the problem, and using the operational matrices to eliminate the derivation operation [3].

In this work, a numerical method based on the operational matrices of fractional derivative for the shifted Legendre polynomials and rational Legendre functions is proposed to solve the fractional diffusion equation with initial-boundary conditions with long time domain. We first give some necessary definitions and mathematical preliminaries of the fractional calculus.

Definition 1.1. [19] A real function $f:\mathbb{R}^+\to\mathbb{R}$, is said to be in the space C_μ, $\mu\in\mathbb{R}$ if there exists a real number $p(>\mu)$ such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C(\mathbb{R}^+)$, and it is said to be in the space C_μ^n if $f^{(n)}\in C_\mu$, $n\in\mathbb{N}$.

Definition 1.2. [19] The Caputo fractional derivative operator of order α ,D^α, is defined on the space C_μ^n by:

$$D^\alpha f(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}dt,\ \alpha>0,\ x>0, \tag{1}$$

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where m is the smallest integer greater than or equal to α.

Recall that for $\alpha \in \mathbb{N}$, the Caputo derivative coincides with the usual differentiation operator of an integer order [19].

The problem considered in this paper is a fractional diffusion equation of the following form:

$$\frac{\partial u(x,t)}{\partial t} = d(x,t)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + s(x,t), \quad 0 < x < b, \ t \geq 0, \tag{2}$$

where the parameter α refers to the fractional order of derivatives with $1 < \alpha \leq 2$, and the function $s(x,t)$ is a source term.

We also consider an initial condition:

$$u(x,0) = u_0(x), \ 0 < x < b, \tag{3}$$

and the boundary conditions:

$$u(0,t) = g_0(t), \quad u(b,t) = g_1(t), \quad t \geq 0. \tag{4}$$

The existence and uniqueness of the solution for such problems are guaranteed in [11,19]. Note that Eq. (2) for $\alpha = 2$ is the classical diffusion equation. For solving the initial-boundary value problems (2)–(4), we first introduce two orthogonal basis for the space of functions on the intervals $[0,b]$ and $[0,+\infty)$, generated by the shifted Legendre polynomials and rational Legendre functions, respectively. Then, we describe some properties of the shifted Legendre polynomials and obtain a new operational matrix of derivative and fractional derivative for these basis functions and also present some useful properties of the rational Legendre functions which are used further in this paper. Next, a new operational matrix method based on the operational matrices of differentiation for the shifted Legendre polynomials and rational Legendre functions is proposed.

This paper is organized as follows: in Section 2, we present the shifted Legendre polynomials and their properties. In Section 3, we obtain the operational matrix for the fractional derivative of the shifted Legendre polynomials. In Section 4, we introduce the rational Legendre functions and also describe some useful properties of these basis functions. In Section 5, we propose a new computational method based on the operational matrices of derivative for the shifted Legendre polynomials and rational Legendre functions. In Section 6, numerical examples are presented to show the effectiveness of the proposed method.

2. Legendre and shifted Legendre polynomials

Let $L_n(z)$ be the Legendre polynomials of degree n which are the eigenfunctions of the singular Sturm–Liouville problem

$$(1-z^2)y'' - 2zy' + n(n+1)y = 0, \ \ n = 0,1,2,\ldots, \quad -1 < z < 1.$$

The Legendre polynomials are orthogonal with respect to the inner product on the interval $[-1,1]$:

$$\int_{-1}^{1} L_m(z)L_n(z)dz = \frac{2}{2n+1}\delta_{mn}$$

where δ_{mn} denotes the Kronecker delta and $L_n(1) = 1$. These polynomials can be determined with the recurrence relation [1]:

$$\begin{aligned} L_0(z) &= 1, \quad L_1(z) = z \\ L_{n+1}(z) &= \frac{2n+1}{n+1} zL_n(z) - \frac{n}{n+1}L_{n-1}(z), \ \ n \geq 1. \end{aligned} \tag{5}$$

In order to use Legendre polynomials on the interval [0,b], we define the shifted Legendre polynomials by presenting the change of variable $z = \frac{2x}{b} - 1$. The shifted Legendre polynomial of degree n on the interval $[0,b]$, $b > 0$, will be denoted by $P_n(x)$ and is given by:

$$P_n(x) := L_n(z) = L_n\left(\frac{2x}{b} - 1\right). \tag{6}$$

Using (5) and (6), we may deduce the recurrence relation for $P_n(x)$ in the following form:

$$\begin{aligned} P_0(x) &= 1, \quad P_1(x) = \frac{2x}{b} - 1, \\ P_{n+1}(x) &= \frac{2n+1}{n+1}\left(\frac{2x}{b} - 1\right)P_n(x) - \frac{n}{n+1}P_{n-1}(x), \ \ n \geq 1. \end{aligned}$$

The expansion of Legendre polynomials $P_n(x)$ of degree n is given by [2]:

$$P_i(x) = \sum_{k=0}^{i} (-1)^{k+i} \frac{(i+k)!\, x^k}{(i-k)!\,(k!)^2\, b^k}. \tag{7}$$

The orthogonality condition for these shifted polynomials on the interval $[0,b]$ is given by:

$$\int_0^b P_m(x)P_n(x)w_s(x)dx = \frac{b}{2n+1}\,\delta_{mn},$$