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Some error estimates on the large jump asymptotic approach for the interface problem ☆

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ABSTRACT

A large jump asymptotic framework is presented for reduction of an elliptic or parabolic problem with strongly discontinuous coefficient jumps to a small number of subproblems with continuous coefficients and independent of the discontinuity amplitude by I. Klapper and T. Shaw (2007) [9]. Based on this frame, error estimates on the approximate asymptotic expansion solution in norms L^2 and H^1 are derived by linear finite element approximations to the corresponding subproblems whose solutions are connected with one by one when the interface is an open curve, and this connection brings the main difficulty in the error analysis. Simultaneously, as the interface is a closed curve, a feasible algorithm is designed according to the characters of the subproblems and the principle of superposition for the differential equation. Numerical experiments are carried on to verify the theoretical result and our new algorithm. Furthermore, a numerical simulation for a radiative heat transfer problem also confirms them with agreeable temperature distributions and small energy conservation error.

1. Introduction

In many applications, a simulation domain is often formed by several materials separated by curves or surfaces from each other, and this often leads to the interface problem consisting of the usual boundary value problem of the diffusion equation, plus jump conditions across the material interface required by pertinent physics. If the interface is smooth enough, then the solution of the interface problem is also very smooth in individual regions where the coefficient is smooth, but due to the jump of the coefficient across the interface, the global regularity is usually low. Because of the low global regularity and the irregular geometry of the interface, achieving accuracy is difficult with standard finite element methods, unless the elements fit with the interface of general shape [1]. It is well known that some popular methods are put out to efficiently solve this type of interface problems, such as the immersed interface method [1–3], the average methods [4,5], the finite element methods [6], the finite difference method and the mixed finite element method [7,8], the asymptotic expansion method [9], and so on. Klapper displayed a large jump asymptotic (LJA) framework for reduction of an elliptic or parabolic problem with strongly discontinuous coefficient jumps to a small number of subproblems, each with continuous coefficients and each independent of the discontinuity amplitude. To our best knowledge, we have found that there are few discussions on error estimates for the LJA approach. Some error estimates

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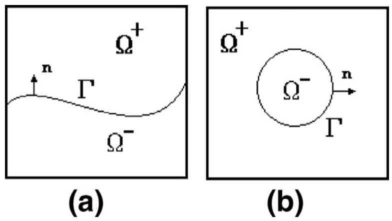

Fig. 1. Two typical cases of the interface: (a) an open curve. (b) A closed curve.

for parabolic interface problems are discussed in [10]. As a matter of fact, error estimates on elliptic interface problems are also worthy of investigating.

In this paper, one innovative idea of our work is that, based on the LJA framework in [9], error estimates on the approximate asymptotic expansion solution in norms L^2 and H^1 are derived by linear finite element approximations to several subproblems when the interface is an open curve. The main difficulty of it not only lies in that the final error estimation for the LJA approach is connected with several subproblems, but also lies in that error estimations of each subproblem are, one by one, relative to the finite element (FE) approximation of another subproblem. It becomes diffiicult, complex and significant because it is greatly different from the estimation of standard FE methods. Another good idea of our work is that, as the interface is a closed curve, a feasible algorithm is designed according to the characters of the subproblems and the principle of superposition for the differential equation. In this case, the solution of the first subproblem is constant, and the second subproblem whose solution is pertinent to this constant, can be decomposed into two basic ones by the principle of superposition for the differential equation, and together with the third subproblem, the constant can be determined. Consequently, the solution of the second subproblem can also be obtained. So similarly do other subproblems. Numerical experiments are carried on to verify the theoretical result and our new algorithm, respectively. Simultaneously, a numerical simulation for a radiative heat conduction problem also confirms them with national temperature distributions and small energy conservation error.

The remainder of this paper is organized as follows. In Section 2, we introduce the interface problem and the LJA method. In Section 3, we derive error estimates in norms L^2 and H^1, and design the algorithm. In Section 4, we describe the radiative heat transfer problem. Finally, we display two numerical examples and a numerical simulation to support our conclusions.

2. The interface problem and the asymptotic expansion method

In this paper, we mainly focus on a type of elliptic equation as follows

$$-\nabla(\beta(\mathbf{x})\nabla u) = f, \tag{1}$$

together with Dirichlet conditions $u = g(\mathbf{x})$ on the boundary of the region $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$, Γ is the interface of Ω^+ and Ω^- (shown as Fig. 1), the discontinuous function

$$\beta = \begin{cases} \alpha^+(\mathbf{x}), & \mathbf{x} = (x_1, x_2) \in \Omega^+, \\ \varepsilon^{-1}\alpha^-(\mathbf{x}), & \mathbf{x} = (x_1, x_2) \in \Omega^-, \end{cases} \tag{2}$$

where $\alpha^+(\mathbf{x})$ and $\alpha^-(\mathbf{x})$ are bounded by some positive constant, respectively. $0 < \varepsilon \le 1$ and the smaller ε is, the larger the jump of the diffusion coefficient is.

The natural jump conditions are as follows

$$[u]|_\Gamma = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right]\bigg|_\Gamma = 0, \tag{3}$$

where

$$[u]|_\Gamma = u^+ - u^-, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right]\bigg|_\Gamma = \alpha^+ u_n^+ - \varepsilon^{-1}\alpha^- u_n^-, \tag{4}$$

u^+, u^- and u_n^+, u_n^- are the solutions on Ω^+, Ω^- and the outer normal derivatives on Γ, respectively.

To be convenient to reveal the core idea, we will derive the approximate subproblems for the above interface problem by two special asymptotic expansions u_ε according to two different types of interfaces.

For the open interface curve, let

$$u_\varepsilon = \begin{cases} u_\varepsilon^+ = u_0^+ + \varepsilon u_1^+ & \mathbf{x} \in \Omega^+, \\ u_\varepsilon^- = u_0^- + \varepsilon u_1^- & \mathbf{x} \in \Omega^-. \end{cases} \tag{5}$$

Substituting (5), (2) into (1) and (3), we can decompose problem (1) into four elliptic subproblems with continuous diffusion coefficients as follows.

$$\begin{cases} -\nabla \cdot (\alpha^- \nabla u_0^-) = 0, & \mathbf{x} \in \Omega^-, \\ u_{0n}^- = 0, & \mathbf{x} \in \Gamma, \\ u_0^- = g, & \mathbf{x} \in \partial\Omega^-. \end{cases} \tag{6}$$