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Optimal quadrature formulas for Cauchy type singular integrals in Sobolev space

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ABSTRACT

In the present paper in $L_2^{(m)}(-1,1)$ space the optimal quadrature formula is constructed for approximate calculation of the Cauchy type singular integral. Explicit formulas for the optimal coefficients are obtained.

1. Introduction: statement of the problem

Many problems of science and engineering are naturally reduced to singular integral equations. Moreover plane problems are reduced to one dimensional singular integral equations (see [10]). The theory of one dimensional singular integral equations is given, for example, in [7,11]. It is known that the solutions of such integral equations are expressed by singular integrals. Therefore approximate calculation of singular integrals with high exactness is actual problem of numerical analysis. For the singular integral of the Cauchy type we consider the following quadrature formula

$$\int_{-1}^{1} \frac{\varphi(x)}{x-t} dx \cong \sum_{\beta=0}^{N} C[\beta]\varphi(x_\beta) \tag{1.1}$$

with the error functional

$$\ell(x) = \frac{\varepsilon_{[-1,1]}(x)}{x-t} - \sum_{\beta=0}^{N} C[\beta]\delta(x-x_\beta), \tag{1.2}$$

where $-1 < t < 1$, $C[\beta]$ are the coefficients, x_β $(\in [-1,1])$ are the nodes, $N \in \mathbb{N}$, $\varepsilon_{[-1,1]}(x)$ is the characteristic function of the interval $[-1,1]$, δ is the Dirac delta function, φ is a function of the space $L_2^{(m)}(-1,1)$. Here $L_2^{(m)}(-1,1)$ is the Sobolev space of functions with a square integrable mth generalized derivative and equipped with the norm

$$\|\varphi|L_2^{(m)}(-1,1)\| = \left\{\int_{-1}^{1} (\varphi^{(m)}(x))^2 dx\right\}^{1/2}$$

and $\{\int_{-1}^{1}(\varphi^{(m)}(x))^2 dx\}^{1/2} < \infty$.

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In order that the error functional (1.2) is defined on the space $L_2^{(m)}(-1,1)$ it is necessary to impose the following conditions (see [20])

$$(\ell(x), x^\alpha) = 0, \quad \alpha = 0, 1, 2, m-1. \tag{1.3}$$

Hence it is clear that for existence of the quadrature formulas of the form (1.1) the condition $N \geq m-1$ has to be met.

The difference

$$(\ell, \varphi) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)dx = \int_{-1}^{1} \frac{\varphi(x)}{x-t}dx - \sum_{\beta=0}^{N} C[\beta]\varphi(x_\beta) \tag{1.4}$$

is called *the error* of the formula (1.1)

By the Cauchy–Schwarz inequality

$$|(\ell, \varphi)| \leq \left\| \varphi | L_2^{(m)} \right\| \cdot \left\| \ell | L_2^{(m)*} \right\|$$

the error (1.4) of the formula (1.1) on functions of the space $L_2^{(m)}(-1,1)$ is reduced to find the norm of the error functional ℓ in the conjugate space $L_2^{(m)*}(-1,1)$.

Obviously the norm of the error functional ℓ depends on the coefficients and the nodes of the quadrature formula (1.1). The problem of finding the minimum of the norm of the error functional ℓ by coefficients and by nodes is called *the S.M. Nikol'skii problem*, and the obtained formula is called *the optimal quadrature formula in the sense of Nikol'skii*. This problem was first considered by Nikol'skii [12], and continued by many authors, see e.g. [13] and references therein. Minimization of the norm of the error functional ℓ by coefficients when the nodes are fixed is called *Sard's problem* and the obtained formula is called *the optimal quadrature formula in the sense of Sard*. First this problem was investigated by Sard [14].

There are several methods of construction of optimal quadrature formulas in the sense of Sard such as the spline method, ϕ—function method (see e.g. [2,9]) and Sobolev's method which is based on construction of discrete analogs of a linear differential operator (see e.g. [19,20]).

The main aim of the present paper is to construct optimal quadrature formulas in the sense of Sard of the form (1.1) in the space $L_2^{(m)}(-1,1)$ by the Sobolev method for approximate integration of the Cauchy type singular integral. This means to find the coefficients $C[\beta]$ which satisfy the following equality

$$\|\mathring{\ell}|L_2^{(m)*}\| = \inf_{C[\beta]} \|\ell|L_2^{(m)*}\|. \tag{1.5}$$

Thus, in order to construct optimal quadrature formulas in the form (1.1) in the sense of Sard we have to consequently solve the following problems.

Problem 1. Find the norm of the error functional (1.2) of the quadrature formula (1.1) in the space $L_2^{(m)*}(-1,1)$.

Problem 2. Find the coefficients $C[\beta]$ which satisfy the equality (1.5).

Many works are devoted to the problem of approximate integration of Cauchy type singular integrals (see, for instance, [1,3,5,8,10,16] and references therein).

The rest of the paper is organized as follows. In Section 2 using a concept of extremal function we find the norm of the error functional (1.2). Section 3 is devoted to a minimization of $\|\ell\|^2$ with respect to the coefficients $C[\beta]$. We obtain a system of linear equations for the coefficients of the optimal quadrature formula in Sard's sense of the form (1.1) in the space $L_2^{(m)}(-1,1)$. Moreover, the existence and uniqueness of the corresponding solution is proved. In Section 4 we give some definitions and known results which we use in the proof of the main results. In Section 5 we give the algorithm for construction of optimal quadrature formulas of the form (1.1). Explicit formulas for coefficients of the optimal quadrature formulas of the form (1.1) are found in Section 6. Finally, in Section 7 some numerical results for the norm of the error functional of the optimal quadrature formulas are given.

2. The extremal function and the expression for the error functional norm

To solve Problem 1, i.e., for finding the norm of the error functional (1.2) in the space $L_2^{(m)}(-1,1)$ a concept of the extremal function is used [20]. The function $\psi_\ell(x)$ is said to be *the extremal function* of the error functional (1.2) if the following equality holds

$$(\ell, \psi_\ell) = \|\ell|L_2^{(m)*}\| \; \|\psi_\ell|L_2^{(m)*}\| \tag{2.1}$$

In the space $L_2^{(m)}$ the extremal function $\psi_\ell(x)$ of a functional $\ell(x)$ is found by S.L. Sobolev [19,20]. This extremal function has the form

$$\psi_\ell = (-1)^m \ell(x) * G_m(x) + P_{m-1}(x), \tag{2.2}$$

where

$$G_m(x) = \frac{|x|^{2m-1}}{2 \cdot (2m-1)!} \tag{2.3}$$