Applied Mathematics and Computation 263 (2015) 392–397

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Fourier transform representation of the generalized hypergeometric functions with applications to the confluent and Gauss hypergeometric functions

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ARTICLE INFO

Keywords:
Hypergeometric and confluent hypergeometric functions
Generalized hypergeometric function
Kampé de Fériet function
Fourier transform
Parseval's identity
Distributional representation

ABSTRACT

We present a Fourier transform representation of the generalized hypergeometric functions. We then use this representation to evaluate integrals of products of two generalized hypergeometric functions. A number of integral identities for confluent and Gauss hypergeometric functions are presented. The results for Euler's gamma function are deduced as special cases.

1. Introduction

The generalized hypergeometric function is defined by [4, p. 155]

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) := \sum_{n=0}^{\infty} \frac{(z)^n (a_1)_n\cdots(a_p)_n}{n!\,(b_1)_n\cdots(b_q)_n}, \tag{1}$$

where $(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$ is the Pochhammer symbol. The series in (1) converges absolutely for all z if $p \le q$ and for $|z| < 1$ if $p = q + 1$, and diverges for all $z \ne 0$ if $p > q + 1$. Many functions that arise in physics and engineering are special cases of generalized hypergeometric functions.

The generalized hypergeometric functions have the following integral representations [4, p. 161]:

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \frac{1}{\Gamma(a_p)}\int_0^{\infty} t^{a_p-1}e^{-t}\;{}_{p-1}F_q((a_{p-1});(b_q);zt)dt \quad (Re(a_p)>0;\, p\le q+1). \tag{2}$$

We will use (2) to find the Fourier transform (FT) [9, Chapter 12] representation of the generalized hypergeometric functions.

Some of generalized hypergeometric functions have special names. For example, ${}_1F_1(a;\, b;\, z)$ also denoted by $\Phi(a;\, b;\, z)$ is called a confluent hypergeometric function (CHF) and satisfies the following, Kummer's differential equation [4, p. 287]:

$$ z\frac{d^2u}{dz^2} + [b-z]\frac{du}{dz} - au = 0. $$

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http://dx.doi.org/10.1016/j.amc.2015.04.083
0096-3003/

The function ${}_2F_1(a, b; c; z)$ is often called the hypergeometric function or Gauss's hypergeometric function (GHF). ${}_2F_1(a, b; c; z)$ some time denoted by $F(a, b; c; z)$. It satisfies the following hypergeometric differential equation [4. p. 260]:

$$z(1-z)\frac{d^2u}{dz^2} + [c - (a+b+1)z]\frac{du}{dz} - abu = 0.$$

Choosing the parameters a, b, c in ${}_1F_1(a; b; z)$ and ${}_2F_1(a, b; c; z)$, appropriately, most of the special functions of mathematical physics such as the Bessel, Leguerre and Legendre functions can be obtained.

The generalized Kampé de Fériet hypergeometric function of two variables is defined by [6, p. 63]

$$F^{p:q;k}_{l:r;s}\left[\begin{matrix}(a_p): & (b_q); & (c_k); \\ (\alpha_l): & (\beta_r); & (\gamma_s);\end{matrix} \quad x, y\right] = \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{m+n}\prod_{j=1}^{q}(b_j)_m\prod_{j=1}^{k}(c_j)_n}{\prod_{j=1}^{l}(\alpha_j)_{m+n}\prod_{j=1}^{r}(\beta_j)_m\prod_{j=1}^{s}(\gamma_j)_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{3}$$

where (λ_p) denotes the array of p parameters $\lambda_1, \lambda_2, \ldots, \lambda_p$. For convergence,

1. $p+q < l+r+1$, $p+k < l+s+1$, $|x| < \infty$, $|y| < \infty$, or
2. $p+q = l+r+1$, $p+k = l+s+1$, and

$$\begin{cases} |x|^{1/(p-l)} + |y|^{1/(p-l)} < 1, & \text{if } p > l, \\ \max\{|x|, |y|\} < 1, & \text{if } p \le l. \end{cases}$$

FT and distributional representations of Euler's gamma and the generalized gamma functions are given in [2] and [8] respectively. In [7] Tassaddiq used FT and distributional representations to investigate the Fermi–Dirac and Bose–Einstein functions. Distributional representation is given by an infinite series of Dirac-delta functions and has led to new results. In this paper we obtain the FT representation of the generalized hypergeometric functions and consequently of confluent and Gauss hypergeometric functions. The results for Euler's gamma function, obtained earlier in [2], are deduced as special cases. We represent generalized Kampé de Fériet's hypergeometric function as an integral of products of two generalized hypergeometric functions.

The plan of this paper is as follows. The FT representation of the generalized hypergeometric functions is given in Sections 2. In Section 3 we present the main result. A number of integral identities for confluent and Gauss hypergeometric functions are presented in Section 4.

2. Fourier representations of the generalized hypergeometric functions

In this section we obtain Fourier and distributional representations of the generalized hypergeometric functions. Substituting $t = e^y$ in (2), we get the FT representation of the generalized hypergeometric function

$$\begin{aligned}&\Gamma(\sigma + i\tau)_pF_q(\sigma + i\tau, a_1, \ldots, a_{p-1}; b_1, \ldots, b_q; z) \\ &= \sqrt{2\pi}\mathcal{F}[e^{\sigma y}\exp(-e^y)_{p-1}F_q(a_1, \ldots, a_{p-1}; b_1, \ldots, b_q; ze^y); \tau] \quad (\sigma > 0; p \le q+1),\end{aligned} \tag{4}$$

where $\mathcal{F}[\varphi; \tau]$ is the FT of φ defined by [9, Chapter 12]

$$\mathcal{F}[\varphi; \tau] := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iy\tau}\varphi(y)dy \quad (\tau \in \mathbb{R} > 0).$$

Remark Using the series expansion

$$\begin{aligned}&e^{\sigma y}\exp(-e^y)_{p-1}F_q(a_1, \ldots, a_{p-1}; b_1, \ldots, b_q; ze^y) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(z)^n(-1)^m(a_1)_n\cdots(a_{p-1})_n}{n!m!(b_1)_n\cdots(b_q)_n}e^{(\sigma+m+n)y} \quad (\sigma > 0; p \le q+1),\end{aligned} \tag{5}$$

and noting that, the FT of the exponential gives a delta function, [9, p. 253]

$$\mathcal{F}[e^{wy}; \tau] = \sqrt{2\pi}\delta(\tau - iw),$$

we can rewrite (4) as a series of delta functions

$$\begin{aligned}&\Gamma(\sigma + i\tau)_pF_q(\sigma + i\tau, a_1, \ldots, a_{p-1}; b_1, \ldots, b_q; z) \\ &= 2\pi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(z)^n(-1)^m(a_1)_n\cdots(a_{p-1})_n}{n!m!(b_1)_n\cdots(b_q)_n}\delta(\tau - (\sigma+n+m)i) \quad (\sigma > 0; p \le q+1).\end{aligned} \tag{6}$$

This is the distributional representation of the generalized hypergeometric function. This representation is only meaningful when defined as the inner product with a function that belongs to a space of the required "good functions". For a detailed discussion of good functions see [3].